AUTOMATED GEOMETRY THEOREM PROVING AND DISCOVERING WITH JAVA GEOMETRY EXPERT (JGEX)

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JAVA GEOMETRY EXPERT (JGEX)

- Similar to other **interactive dynamic geometry** system
- Can make geometrical theorem **formal proofs**
- Developed on 1980 by Shang Chou, Xiao Shan Gao and Zheng Ye
- One of the most **complete programs** in the field

JGEX FEATURES

- Tools for **designing** geometric figures
- More **formal design rules**
- It has a **core of 45 rules** used to make proofs, most of which are **common theorems** of Euclid Geometry
- There are four different **proving methods**:
    - **Deductive Database**
    - Full-angles method
    - Groebner Basis
    - Wu's Method

FIXPOINT

- **Library** of figure **properties** that is constructed to **enable proof**
- Contains from dozens to thousands of properties that are used as the Deductive Database method’s **starting point**
- One of the most useful capabilities of this software
- Even if the program is not able to prove a theorem, it enables students and teachers to **evaluate claims**

MATHEMATICAL PROOFS

- No accurate definition
- **Mathematical procedure** to solve a problem
- The proof concept is invented by Ancient Greeks
- Two big categories:
  - Formal
  - Informal

FORMAL/INFORMAL PROOFS

Formal

- Typical procedures
- Direct result of logic rules
- Applied on axiom systems
- Mainly used in modern applications in Informatics (i.e. automation)

Informal

- Utilize deductive rules
- Steps could be skipped
- Approaches can be generated ex *nihilo*
- We usually find ideas that can't be extracted directly from a formal procedure

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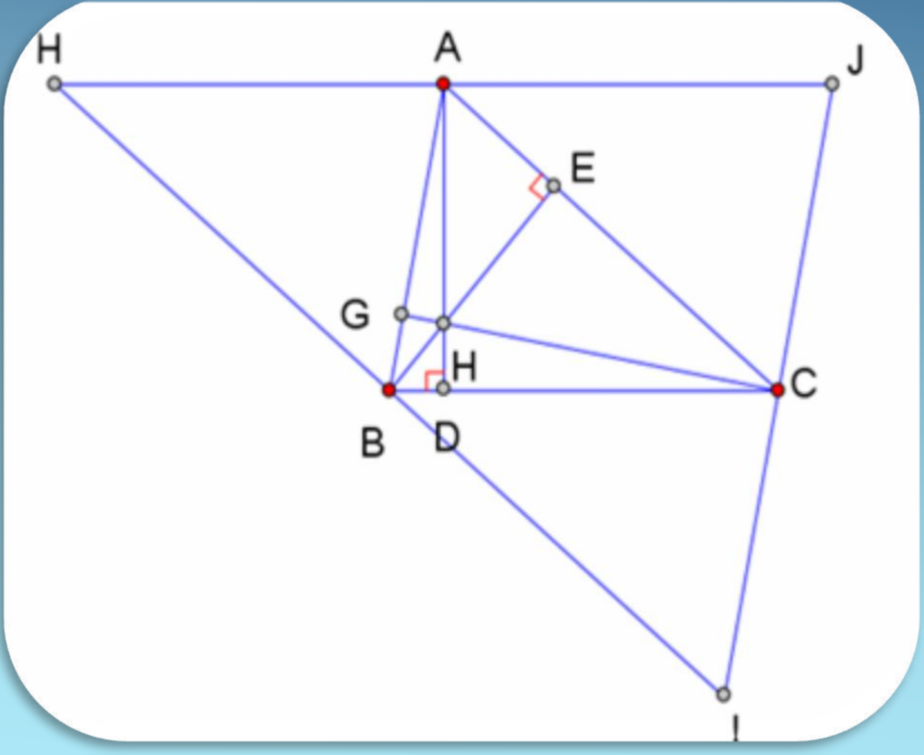
CLASSICAL GEOMETRIC PROOF (GAUSS)



Show that the three altitudes of a triangle are concurrent

Constructing triangle HJI:

HJ, JI and HI parallel to BC, AB and AC

- The points A,B,C of triangle ABC are the midpoints of HJ, HI and JI
- Then AD, EB and GC are the perp-bisect lines of HJ, HI, JI in triangle HJI

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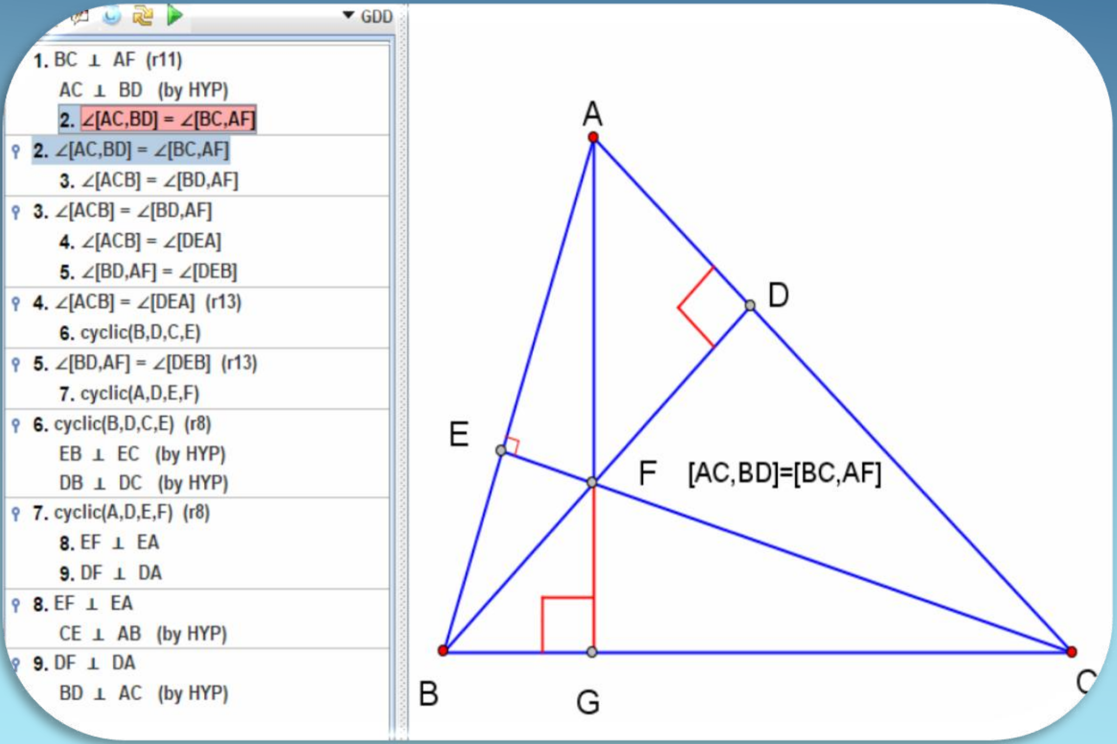
MACHINE PROOF

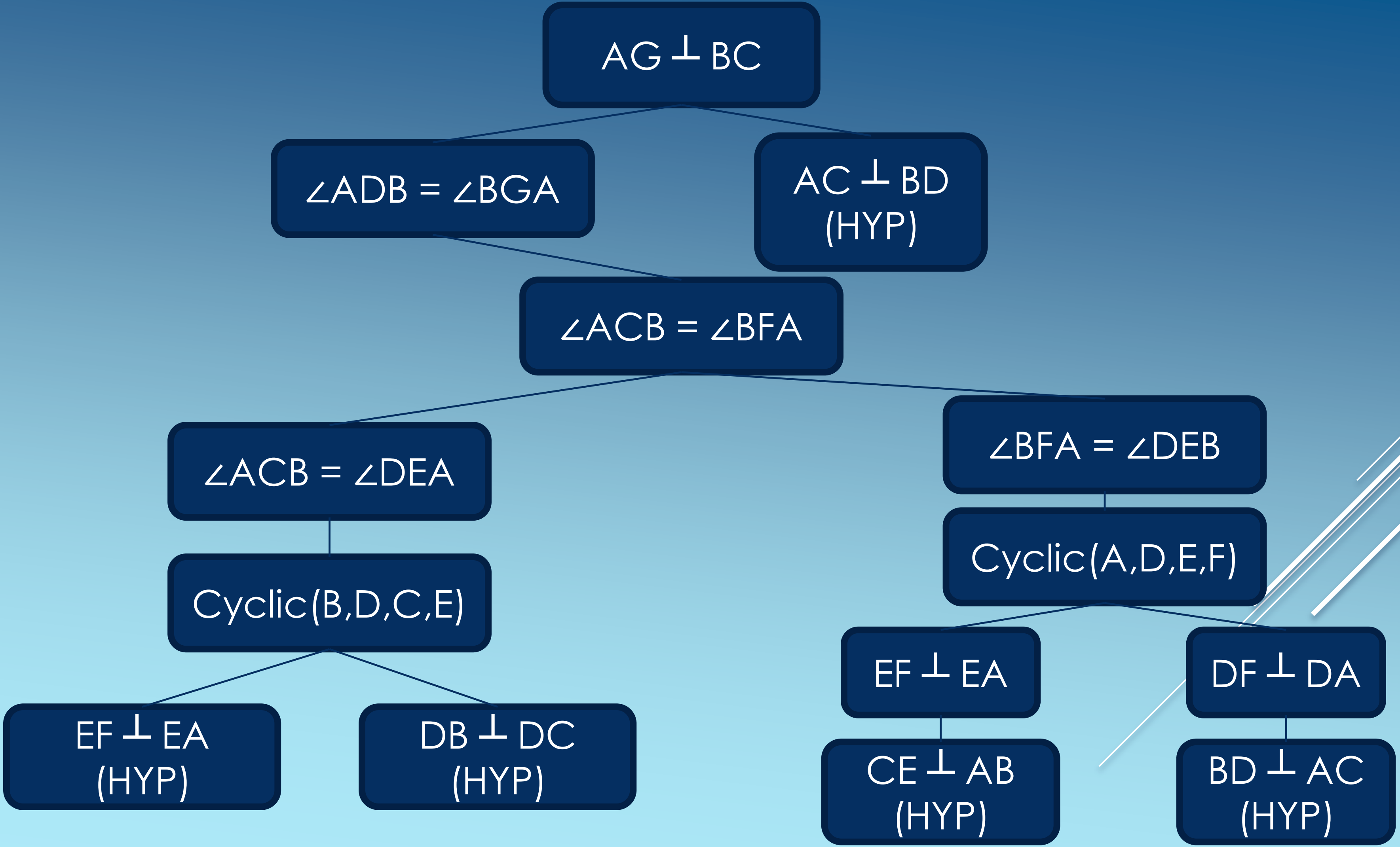
If BD is perpendicular to AC and CE perpendicular to AB, then AF is perpendicular to BC



The main goal is to prove the equality between the angles [AC,BD] and [BC,AF] and that AC is perpendicular to BD

Database-Fixpoint

- collinear point sets: 8
- similiar triangles: 19
- perpendicular lines: 28
- ratio segments: 110
- circles: 6
- congruent angles: 140



AG ⊥ BC
∠ADB = ∠BGA
AC ⊥ BD (HYP)
∠ACB = ∠BFA
∠ACB = ∠DEA
∠BFA = ∠DEB
Cyclic(B,D,C,E)
Cyclic(A,D,E,F)
EF ⊥ EA (HYP)
DB ⊥ DC (HYP)
EF ⊥ EA
DF ⊥ DA
CE ⊥ AB (HYP)
BD ⊥ AC (HYP)

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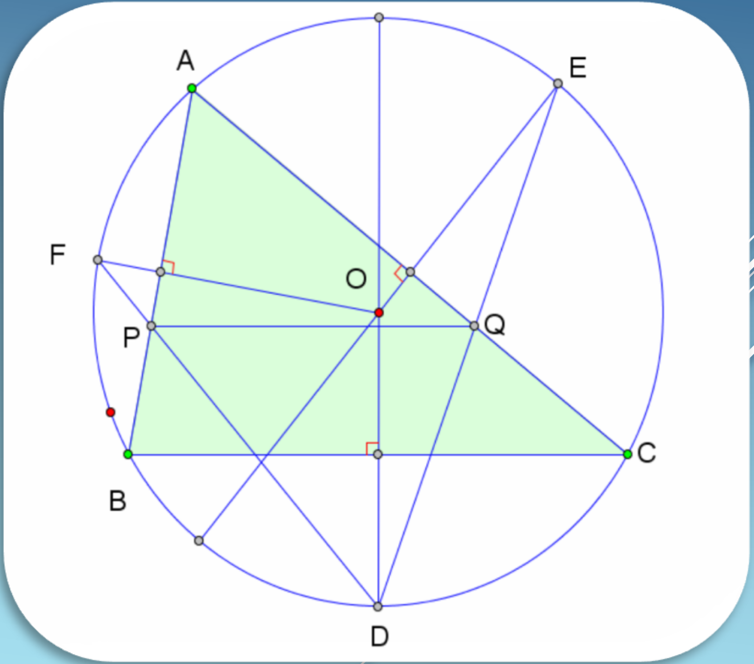
Let **O** be the **circumcircle** of **triangle ABC** and let **D**, **E** and **F** be the midpoints of those arcs BC, AC, AB of **O**, that do not contain points A, B, C respectively.

If:

1)P is the intersection of AB and DF and

2)Q is the intersection of AC and DE,

prove that PQ is parallel to BC.



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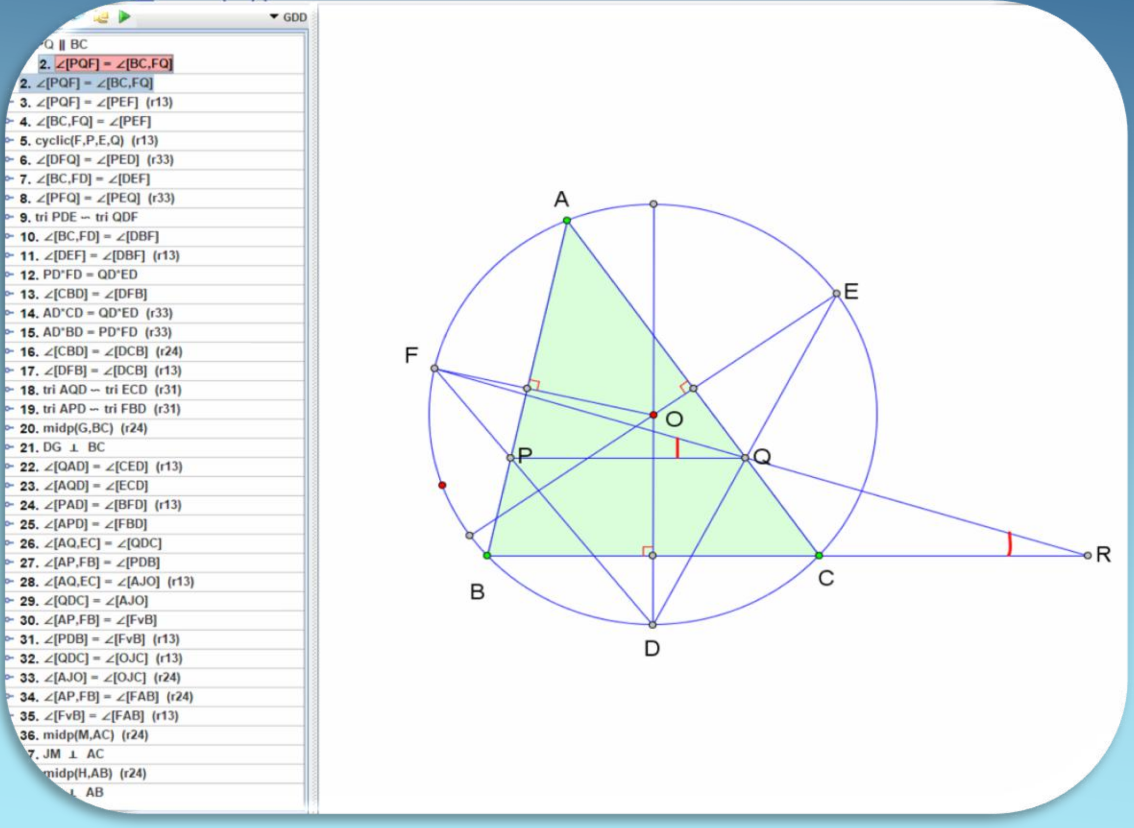
MACHINE PROOF



The main goal is to prove the angle equality <u>[PQF]=[BC,FQ]</u>

Database - fixpoint

- collinear point sets: 8
- similiar triangles: 19
- parallel lines: 9
- congruent triangles: 31
- perpendicular lines: 28
- ratio segments: 110
- midpoints: 5
- circles: 6
- congruent segments: 11
- congruent angles: 140

USEFUL GEOMETRIC RULES FOR MACHINE PROOF

A **geometric rules:** - p1,...,pk are geometry predicates

$$\forall x \left[\left(p_1(x) \wedge p_2(x) \wedge \cdots \wedge p_k(x) \right) \Rightarrow Q(x) \right]$$

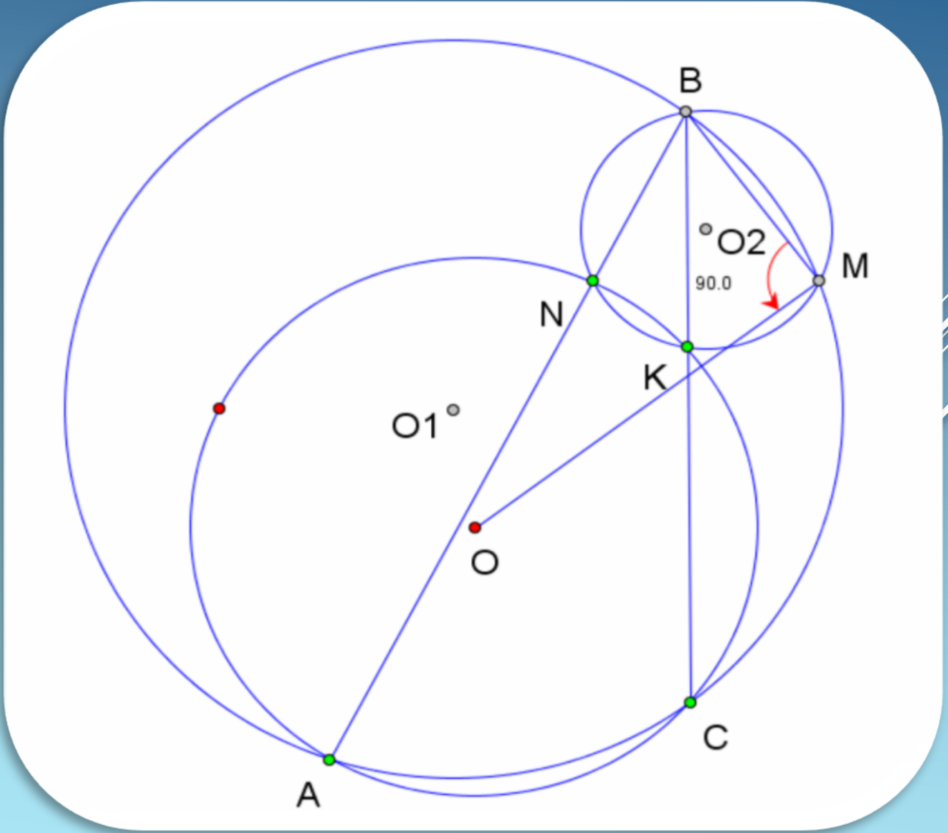
One of the central **geometric concepts** is the **full-angle**

- The full angle ∠[u, v] is the angle **from line u to line v**
- Two full angles ∠[l, m] and ∠[u, v] are equal **if a rotation R exists such that R(l) // u ∧ R(m) // v**
- The introduction of full-angles greatly <u>**simplifies the predicate of the angle congruence**</u>

INTERNATIONAL MATHEMATICS OLYMPIAD 1985

- Let **A,C,K** and **N** be four points on a circle.
- **B** is the **intersection** of AN and CK.
- **M** is the **intersection** of the circumcircle of triangles BKN and BAC.

Show that BM is perpendicular to MO.



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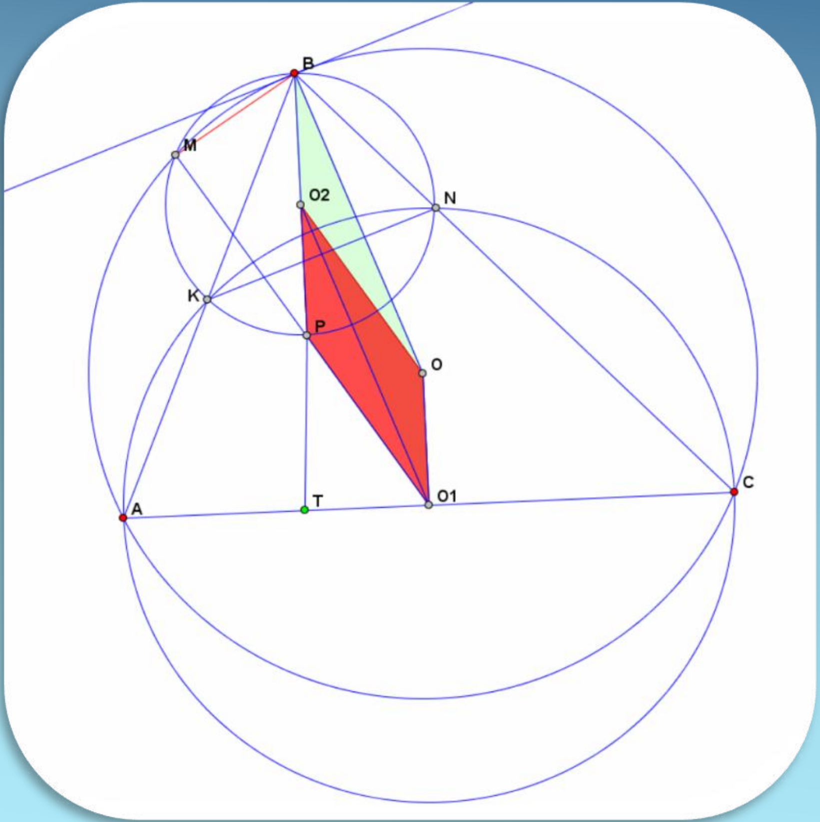
CLASSICAL GEOMETRIC PROOF



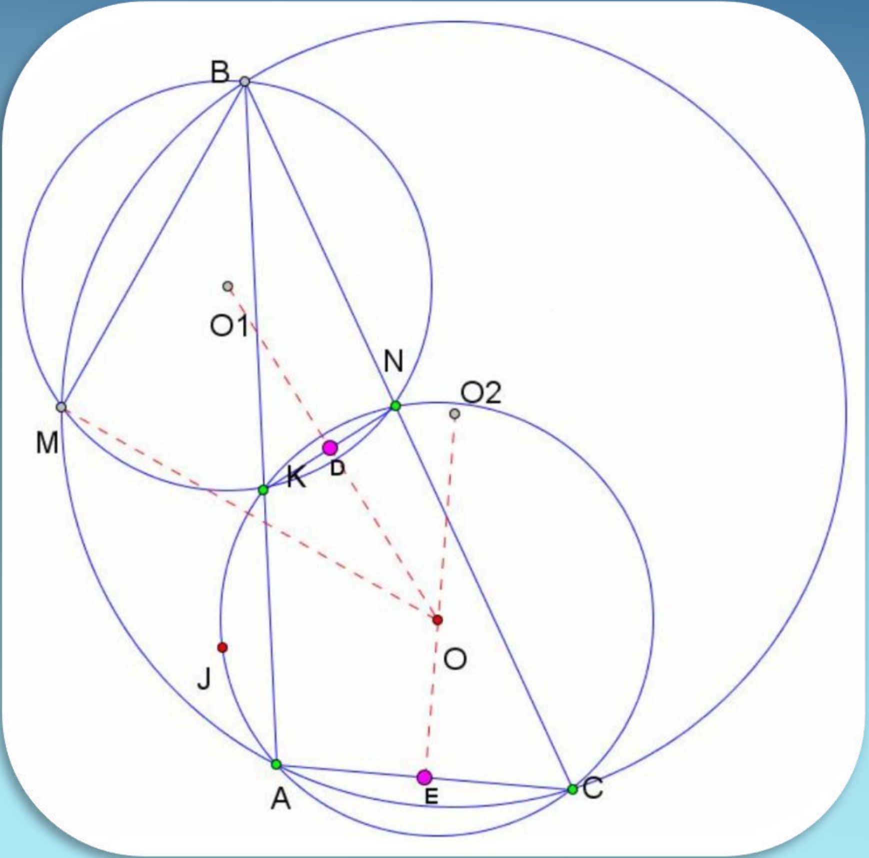
- ❑ Constructing Auxiliary Points P, T and H
- ❑ Constructing line e parallel to KN through B

P, T = the second points of intersection of the circumcircle of BKN and the lines BO2 and AC

H = the second point of intersection of O1P and the circumcircle of BKN

- The opposite sides of OBO2O1 are parallel
- The quadrangle OO1PO2 is a parallelogram
- The opposite sides of O1OO2H are parallel

INTERNATIONAL MATHEMATICS OLYMPIAD 1985
MACHINE PROOF



Auxiliary point D as the intersection of O_1O and KN

Prove that the angles [BMO] and [O1O2B] are equal and O1O2 is perpendicular to BM.

Database – fixpoint:

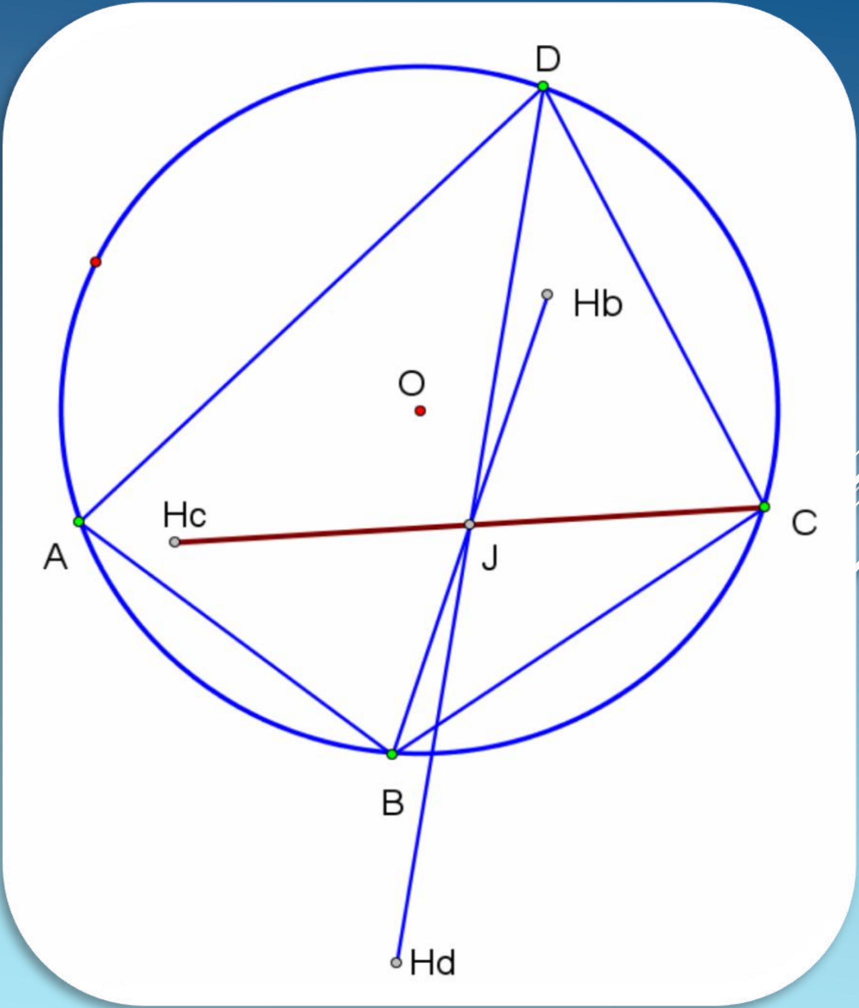
- collinear point sets: 2
- similiar triangles: 9
- congruent triangles: 3
- perpendicular lines: 3
- ratio segments: 33
- circles: 6
- congruent segments: 3
- congruent angles: 59

THEOREM 3

Let A,B,C,D be four points on a circle.
If:

- **Ha is the orthocenter** of triangle BCD,
- **Hb is the orthocenter** of triangle ACD,
- **Hc is the orthocenter** of triangle ABD and
- **Hd is the orthocenter** of triangle ABC,

prove that J is the intersection of AHa, BHb, CHc and DHd.



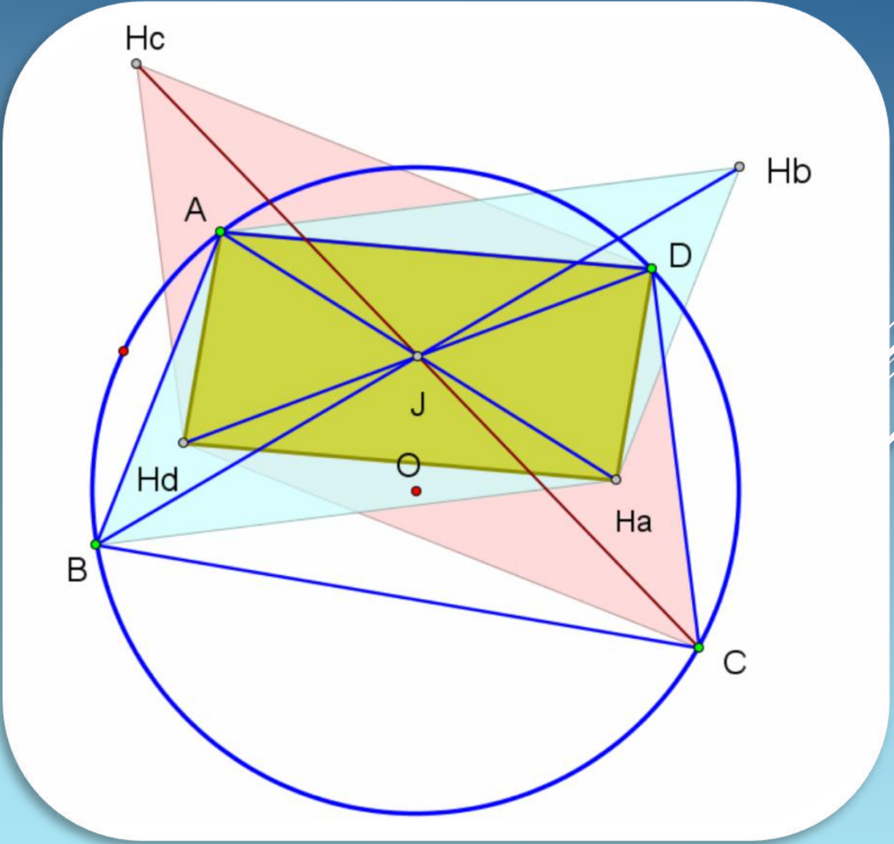
THEOREM 3 : CLASSICAL PROOF

The classical geometric proof is simple but we need construct three new objects:

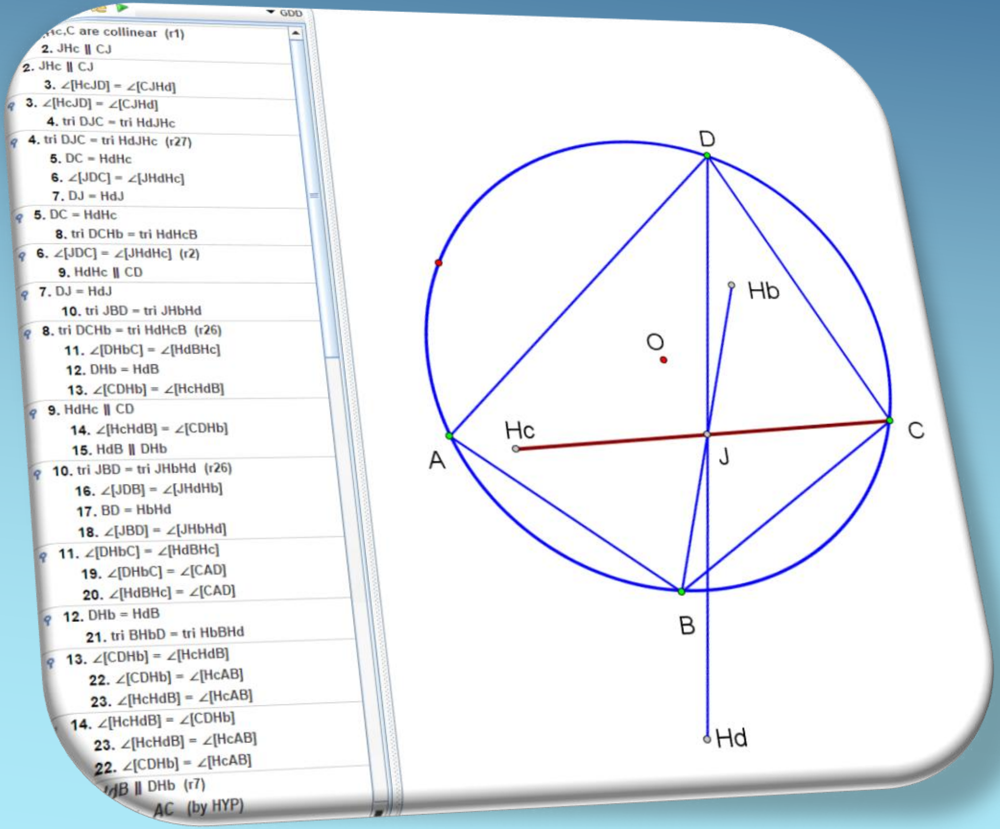
the parallelogrammes AHdHaD, HcHdCD and BHaHbA.

The proof consist to observe that the diagonals:

AHa and, DHd, intersect in J point and this point J is also the **center of symmetry** of the parallelogrammes HcHdCD and BHaHbA.



THEOREM 3
MACHINE PROOF



Let DHd and BHb interesect in J

The main goal is to prove that JHc is parallel to CJ.

Database – fixpoint:

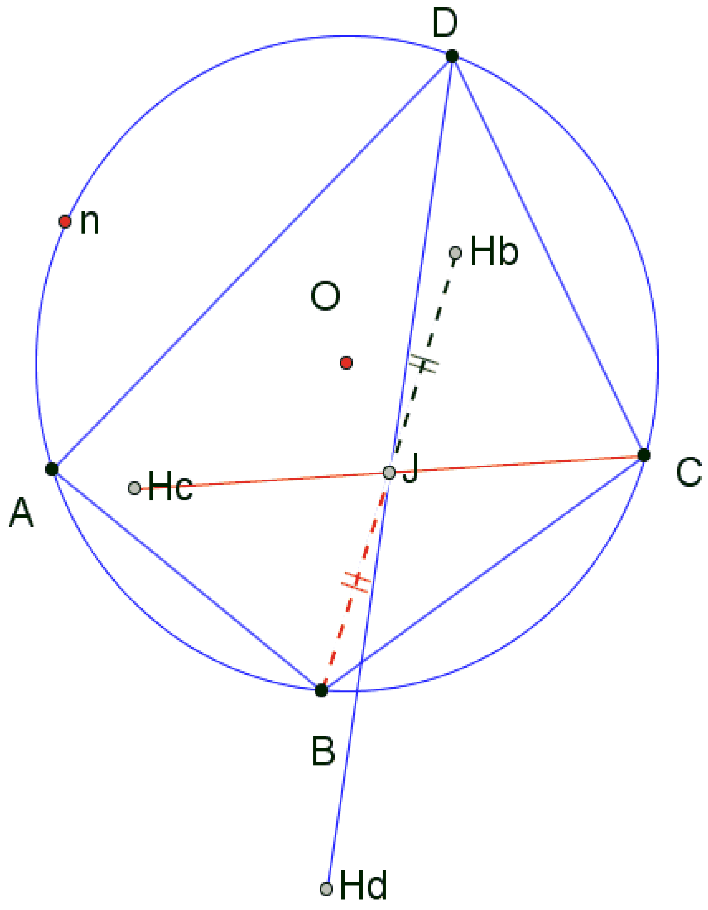
- collinear point sets: 3
- similiar triangles: 9
- parallel lines: 6
- congruent triangles: 16
- perpendicular lines: 12
- ratio segments: 10
- midpoints: 3
- circles: 4
- congruent segments: 10
- congruent angles: 55

GDD

Fixpoint

- lines (3)
- parallel lines (6)
  - plines[D,Hb; B,Hd]
  - plines[C,Hb; B,Hc]
  - plines[C,Hd; D,Hc]
  - plines[B,D; Hb,Hd]
  - plines[B,C; Hb,Hc]
  - plines[Hd,Hc; C,D]
- perpendicular lines (12)
- midpoints (3)
- circles (4)
  - circle[O, nABCD]
  - circle[ADHbHc]
  - circle[ACHbHd]
  - circle[ABHdHc]
- congruent segments (10)
- congruent angles (55)
- similiar triangles (9)
- congruent triangles (16)
- ratio segments (10)

Thm | F | D | A | M | Fix



THANK YOU FOR YOUR ATTENTION!

References:

1) Chou, S. C., *Mechanical Geometry Theorem Proving*, D. Reidel Publishing Company, Dordrecht, Netherlands, 1988.
2) Chou, S., C., Gao X. S. and Zhang J. Z., *Machine Proofs in Geometry*, World Scientific, 1994.
3) Wu Wen-tsun, *Mechanical Theorem Proving in Geometries*, Springer Verlag, Texts and Monographs, 1994.

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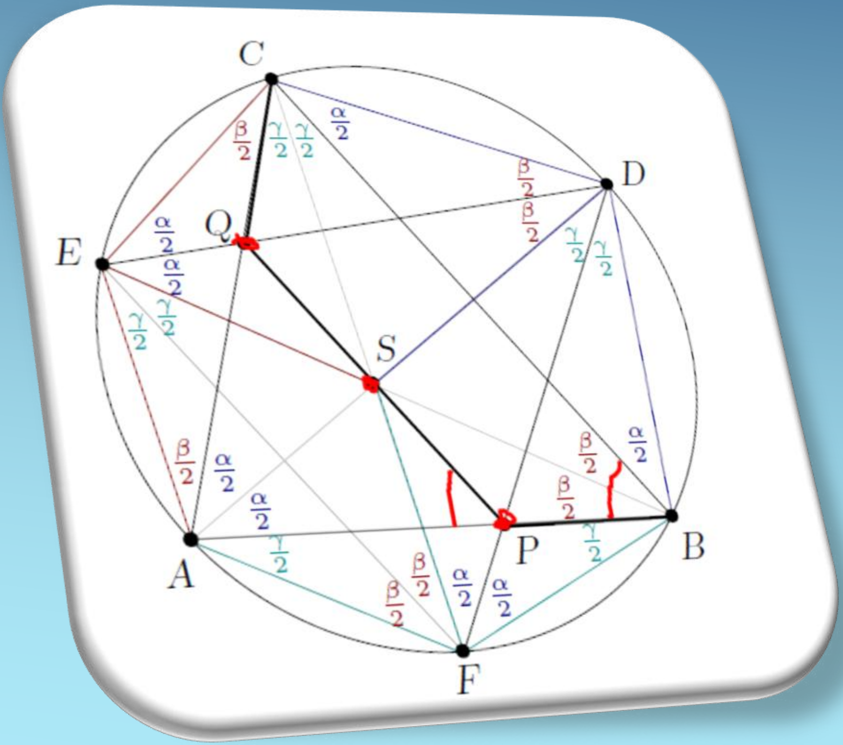
CLASSICAL GEOMETRIC PROOF



Constructing Auxiliary Point. **S**

S = the center of the incircle of ABC

- The points P, Q and S are **collinear points**
- Then, **angle SPA = angle CBA**

EXAMPLE (FULL-ANGLES)

If using ordinary angles, we need to specify the relation among 8 angles and we need to **use order relation to distinguish the cases**.

For instance, if point B, D are on the same side of line PQ and points P, C are on different sides of line AB, then AB//CD ⇔ ∠ PEB = ∠PFD

This rule is very difficult to use and may lead **to branchings during the deduction**