The 54th International Mathematical Olympiad: Problems and Solutions

Day 1 (July 23th, 2013)

Problem 1

Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \dots, m_k$ such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Hide solution

We will prove the statement using induction on k. More precisely, we will prove the following:

For $k=1$, we can take $m_1=n$, and the required equality is trivial.

Assume now that statement holds for a positive integer k. We want to prove that for every integer n there exists positive integers $m_1, \dots, m_{k+1}$ that satisfy

$$1+\frac{2^{k+1}-1}{n}=\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_{k+1}}\right).$$

We will distinguish the cases of odd and even n.

- **Case 1.** n is odd. Then $\frac{n+1}{2}\in\mathbb{N}$ and according to inductional hypothesis there are integers $m_1, \dots, m_k$ such that

  $$1+\frac{2^k-1}{\frac{n+1}{2}}=\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

  By taking $m_{k+1}=n$ we obtain:

  $$\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_{k+1}}\right)=\left(1+\frac{2^k-1}{\frac{n+1}{2}}\right)\cdot\left(1+\frac{1}{n}\right)=\left(1+\frac{2^{k+1}-2}{n+1}\right)\cdot\frac{n+1}{n}=\frac{n+1+2^{k+1}-2}{n}=1+\frac{2^{k+1}-1}{n}.$$

- **Case 2.** n is even. Then $\frac{n}{2}\in\mathbb{N}$ and according to inductional hypothesis there are integers $m_1, \dots, m_k$ such that

  $$1+\frac{2^k-1}{\frac{n}{2}}=\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

  By taking $m_{k+1}=2^{k+1}+n-2$ we obtain:

  $$\left(1+\frac{1}{m_1}\right)\cdots\left(1+\frac{1}{m_{k+1}}\right)=\left(1+\frac{2^k-1}{\frac{n}{2}}\right)\cdot\left(1+\frac{1}{2^{k+1}+n-2}\right)=\left(1+\frac{2^{k+1}-2}{n}\right)\cdot\left(1+\frac{1}{2^{k+1}+n-2}\right)$$
  $$=\frac{2^{k+1}+n-2}{n}\cdot\frac{2^{k+1}+n-2+1}{2^{k+1}+n-2}=1+\frac{2^{k+1}-1}{n}.$$

Problem 2

Given 2013 red and 2014 blue points in the plane, assume that no three of the given points lie on a line. A partition of the plane is called *perfect* if no region contains points of different colors. Determine the smallest number k for which a perfect partition can always be achieved by drawing k straight lines.

Hide solution

We will prove that 2013 is the smallest value for k for which one can guarantee the perfect partition.

We will first show that there is a configuration of points for which 2013 lines are necessary to achieve the perfect partition. Consider the regular 4027-gon $A_1 A_2 \cdots A_{4027}$ inscribed in the unit circle. Assume that the vertices A_{2k+1} are red for $k \in \{0, 1, \ldots, 2013\}$ and that the other vertices are blue. For each $j \in \{1, 2, \ldots, 4026\}$ the shorter arc $A_j A_{j+1}$ has to intersect one of the lines of the partition. Hence there are 4026 points of intersections of the unit circle with the lines of the partition. Since each line can contain at most 2 of the points, we must have at least 2013 lines in the partition.

We will now prove that it is always possible to form a perfect partition using 2013 lines. Assume that there are 2013 red and 2014 blue points. Let $W_1 \ldots W_k$ be the convex hull of these points.

If any point of this convex hull is red, say W_1, then there is a line m that separates the plane into two regions one of which contains W_1 only. Let $\{R_1, \ldots, R_{2012}\}$ be the set of red points other than W_1. For each $j \in \{1, 2, \ldots, 1006\}$ we consider the line $R_{2j-1}R_{2j}$. There are two lines l_j and l'_j parallel to $R_{2j-1}R_{2j}$ such that the region between l_j and l'_j does not contain any points other than R_{2j-1} and R_{2j}. The lines m, l_1, l_1, l_2, l'_2, $\ldots$, l_{1006}, l'_{1006} generate a perfect partition.

If all of the points W_1, $\ldots$, W_k are blue, then there is a line p parallel to $W_1 W_2$ that separates the plane into two regions: one containing only the points W_1 and W_2. Let us denote by B_1, $\ldots$, B_{2012} the remaining blue points. For each $j \in \{1, 2, \ldots, 1006\}$ we consider the line $B_{2j-1}B_{2j}$. There are two lines l_j and l'_j parallel to $B_{2j-1}B_{2j}$ such that the region between l_j and l'_j does not contain any points other than B_{2j-1} and B_{2j}. The lines p, l_1, l_1, l_2, l'_2, $\ldots$, l_{1006}, l'_{1006} generate a perfect partition.

Problem 3

Let A_1, B_1, and C_1 be the points at which the excircles touch the sides BC, CA, and AB of the triangle ABC. Prove that if the circumcenter of $\triangle A_1B_1C_1$ belongs to the circumcircle of $\triangle ABC$, then one of the angles of $\triangle ABC$ is 90°.

Hide solution

Denote by a, b, and c the lengths of the sides BC, CA, and AB. Let $s=\frac{a+b+c}{2}$. Denote by A_2, B_2, and C_2 the points of tangency of the incircle with the sides BC, CA, and AB. Let k_A, k_B, and k_C be the circumcircles corresponding to A, B, and C, and let S_A, S_B, and S_C be their centers, respectively. Then we have

$$AC_2=AB_2=BC_1=CB_1=s-a,\qquad BA_2=BC_2=CA_1=AC_1=s-b,\qquad CA_2=CB_2=BA_1=AB_1=s-c.$$

Since the circumcenter of $\triangle A_1B_1C_1$ belongs to the circumcircle of $\triangle ABC$, one of the angles of $\triangle A_1B_1C_1$ has to be obtuse. Assume that $\angle B_1A_1C_1>90^\circ$. Then the circumcenter of $\triangle A_1B_1C_1$ and the point A belong to the same arc BC of the circumcircle of $\triangle ABC$.

Let M be the midpoint of the arc BC that contains A. From $MB=MC$, $\angle MBA=\angle MCA$, and $BC_1=CB_1$ we conclude that $\triangle MBC_1\cong\triangle MCB_1$ and $MB_1=MC_1$. Therefore, M is the circumcenter of $\triangle A_1B_1C_1$.

Since $BA_2=CA_1$, we conclude that $MA_2=MA_1$ and consequently that A_2 belongs to the circumcircle of $\triangle A_1B_1C_1$. let B_c be the point of tangency of AB with the circle k_B. Since M is the midpoint of S_CS_B and $S_CC_1||S_BB_c$ we conclude that K belongs to the bisector of the segment C_1B_c. This implies that B_c belongs to the circumcircle of $\triangle A_1B_1C_1$.

We now use the power of the point B with respect to the circumcircle of $\triangle A_1B_1C_1$. We have $BA_2\cdot BA_1=BC_1\cdot BB_c$. Since $BB_c=s$, $BC_1=s-a$, $BA_2=s-b$ and $BA_1=s-c$ we conclude $s(s-a)=(s-b)(s-c)$ which is equivalent to $a^2=b^2+c^2$. According to Pythagoras' theorem this implies that $\angle CAB=90^\circ$.

Day 2 (July 24th, 2013)

Problem 4

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC between B and C. The points M and N are the feet of perpendiculars from B and C, respectively. Let ω_1 be the circumcircle of $\triangle BWN$, and let X be the point such that WX is a diameter of ω_1. Let ω_2 be the circumcircle of $\triangle CWM$, and let Y be the point such that WY is a diameter of ω_2. Prove that the points X, Y, and H are collinear.

Hide solution

Let Z be the other point of intersection of ω_1 and ω_2. We will prove that X, Z, and H are collinear. In the same way we obtain that Y, Z, and H are collinear. From $\angle NZW = 180^\circ - \angle ABC$ and $\angle MZW = 180^\circ - \angle ACB$ we conclude that $\angle NZM = 180^\circ - \angle BAC$ which means that Z belongs to the circumcircle of $\triangle NMA$.

Therefore $\angle AZN = \angle AMN$, and since $BCMN$ is cyclic we have $\angle AMN = \angle ABC$. Thus $\angle AZN + \angle NZW = 180^\circ$ and the points A, Z, and W lie on a line.

Let P be the foot of perpendicular from A to BC. Then $BEHN$ is cyclic hence $AH \cdot AE = AN \cdot AB$. We also have $AZ \cdot AW = AN \cdot AB = AH \cdot AP$, therefore $EWZH$ is cyclic and $\angle HZW = \angle HEW = 90^\circ$, thus the line ZH intersects ω_1 at X.

In a similar way we obtain that Y, Z, and H are collinear, and the statement is proved.

Problem 5

Let $\mathbb{Q}_+$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_+ \to \mathbb{R}$ be a function that satisfies the following three conditions:

- **(i)** $f(x)f(y) \geq f(xy)$ for all $x, y \in \mathbb{Q}_+$,
- **(ii)** $f(x+y) \geq f(x) + f(y)$ for all $x, y \in \mathbb{Q}_+$,
- **(iii)** There exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_+$.

Hide solution

From $a = f(a) = f(a \cdot 1) \leq f(a) \cdot f(1) = a \cdot f(1)$ we conclude that $f(1) \geq 1$. For any positive integer n we have $f(n) = f(1 + n - 1) \geq f(1) + f(n-1) \geq 1 + f(n-1)$, and an argument with induction implies that $f(n) \geq n$ for all $n \in \mathbb{N}$.

If $m, n \in \mathbb{N}$ we have $m \leq f(m) \leq f(\frac{m}{n}) \cdot f(n)$ which implies that $f(\frac{m}{n}) > 0$. Therefore, the codomain of f is $\mathbb{Q}_+$ and the condition (iii) implies that f is increasing.

Since $f(a) = a$ we have $a^2 = f(a) \cdot f(a) \geq f(a^2)$ and the principle of mathematical induction implies that for each $k \in \mathbb{N}$ we have $f(a^k) \leq a^k$. Clearly, $f(na) \geq nf(a) = na$.

Assume that for some $l \in \mathbb{N}$ we have $f(la) - la = \alpha > 0$. If N is any integer greater that $\frac{a}{\alpha}$ we conclude that $f(Nla) \geq Nf(la) \geq Nla + N\alpha > Nla + a$. There exists $k \in \mathbb{N}$ such that $\lfloor a^k \rfloor > Nl$. The condition (ii) implies:

$$a^{k+1} \geq f\left(a^{k+1}a\right) \geq f\left(\lfloor a^k \rfloor a\right) \geq f\left(\left(\lfloor a^k \rfloor - Nl\right)a\right) + f(Nla)$$
$$> \left(\lfloor a^k \rfloor - Nl\right)a + Nla + a = \lfloor a^k \rfloor a + a.$$

This is a contradiction, hence $f(la) = la$ for all $l \in \mathbb{N}$.

There are integers p and q such that $a = \frac{p}{q}$. Hence $f(d\,\frac{p}{q}) = d\,\frac{p}{q}$ for all $d \in \mathbb{N}$ and setting $d = kq$ gives us $f(kp) = kp$ for all $k \in \mathbb{N}$. From the second equation we have $kp = f(kp) \geq f(k) + (p-1)f(k) \geq 2f(k) + (p-2)f(k) \geq \cdots \geq pf(k)$ hence $f(k) \leq k$. Together with already established inequality $f(k) \geq k$ we conclude that $f(k) = k$ for all $k \in \mathbb{N}$.

Assume that If for some $x \in \mathbb{Q}_+$ there exists $\beta > 0$ such that $f(x) - x = \beta$. Let N be an integer such that $Nx \in \mathbb{N}$. Then we have $Nx = f(Nx) \geq Nf(x) = N(x + \beta) = Nx + N\beta$. This is a contradiction. Thus $f(x) = x$ for all $x \in \mathbb{Q}_+$.

Problem 6

Given an integer $n \geq 3$, assume that $n+1$ equally spaced points are marked on a circle. Consider all labelings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once. Two labelings are considered the same if one can be obtained from the other by a rotation of the circle. A labeling is *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining points labeled a and d does not intersect the chord joining points labeled b and c.

Let M be the number of beautiful labelings and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.

Hide solution

Assume that $a_0, a_1, \ldots, a_n$ are the labels at the vertices of the $n+1$-gon in counter-clockwise order. We may assume that $a_0 = 0$ since rotations preserve labelings. We say that a quadruple (a, b, c, d) is *balanced* if $a + c = b + d$. A labeling is beautiful if and only if no four vertices form balanced quadruple when arranged in counter-clockwise orientation.

Given a beautiful labeling $a_0, \ldots, a_n$ with $a_0 = 0$, assume that $a_1 = x$.

Lemma If $k = \lfloor \frac{n}{x} \rfloor$, then the equality $a_i = ix$ holds for $0 \leq i \leq k$.

Proof of Lemma. Assume the contrary: there is $i < k$ such that $a_i \neq ix$. We may assume that $a_q = qx$ for $q \leq i-1$. If $a_i > x$, then there is $j > i$ such that $a_j = a_i - x$, and (a_0, a_1, a_i, a_j) is a balanced quadruple.

Assume that $a_i < x$. Assume that $a_j = ix$ for some $j > i$. Since $(i-1)x + a_i \notin \{a_0, \ldots, a_i, a_j\}$ there is l such that $a_l = (i-1)x + a_i$. If $l < j$ then the configuration wouldn't be beautiful as $(a_1, a_i, a_l, a_j) = (x, a_i, (i-1)x + a_i, ix)$ would form a balanced quadruple. Thus $l > j$.

Assume that m is the index such that $a_m = ix - a_i$. Clearly $ix - a_i \notin \{a_0, \ldots, a_i, a_j\}$. If $m > j$ then (a_0, a_i, a_j, a_m) would form a balanced quadruple, hence $m < j$.

However, the quadruple $(a_{i-1}, a_m, a_j, a_l) = ((i-1)x, ix - a_i, ix, (i-1)x + a_i)$ is balanced. This contradiction completes the proof of the lemma. □

Assume that $a_{k+1} = y$. Let p be the smallest integer such that $px - y > n$. Let $m = px - y$. For $0 \leq t < m$, let us denote by b_t the remainder of tx modulo m. We will now prove that $a_0, a_1, \ldots, a_n$ is the subsequence of the sequence $b_0, b_1, \ldots, b_m$ consisting of those elements that belong to $\{0, 1, \ldots, n\}$.

It suffices to prove that $b_0, \ldots, b_m$ is the unique beautiful sequence that satisfies $b_j = jx$ for $j = 0, \ldots, k$, and $b_{k+1} = y$. Assume that $c_0, \ldots, c_m$ is a beautiful sequence such that $c_i = b_i$ for $0 \leq i \leq k+1$ and let j be the smallest integer such that $c_j \neq b_j$.

First of all, we must have $c_j < x$, as otherwise, the number $c_j - x$ would not be an element of $\{c_0, \ldots, c_j\}$ and the sequence $(c_0, c_1, c_j, c_j - x)$ would be balanced.

Since the sequence (c_j) is nice, the label $c_k + c_j - c_{k+1}$ is either negative or appears in the set $\{c_0, \ldots, c_j\}$. Since
$c_k + c_j - c_{k+1} = kx + c_j - px + m = (k-p)x + c_j + m$

This means that $kx + c_j - y = x + c_j + m - px$ belongs to the set $c_j + m - (p-1)x \in \{c_0, \ldots, c_j\}$. This is contradicts our assumption on c_j.

For $x = 1$, we obtain the beautiful sequence $a_n = n$, and every other beautiful sequence is uniquely determined by the pair (x, y) of relatively prime integers smaller than or equal to n such that $y < x$. Denote by S the set of all such pairs, and let $f : S \to \{1, 2, \ldots, n\}^2$ be the function defined as $f(x, y) = (x, x - y)$. Clearly f is the bijection of S to the set of all ordered pairs (a, b) of relatively prime integers smaller than n such that $a + b \leq n$.

Thus $M = N + 1$.