European Student Conference in Mathematics
EUROMATH 2011

„Intersection points of the diagonals of a regular polygon”

AUTHOR:
Ivan Dinkov Gerganov

SCIENTIFIC SUPERVISOR:
Valcho Milchev

Summary

The present project „Intersection points of the diagonals of a regular polygon" is a research about the connection between the number of the intersection points of the diagonals of a regular n-gon inside it and the number of unordered breaks of natural numbers in 3 addends.

Connection between these two different combinatorial problems is found in the project. A formula about the number of intersection points of $12k+2$, $12k-2$, $12k+4$ and $12k-4$-gon inside it is derived by the theory of breaks of numbers: $S_n = \binom{n}{4} - \frac{5n^3 - 45n^2 + 70n - 24}{24}$.

Here it is made a research about the relation between the number of the intersection points of the diagonals of a regular n-gon inside it and the number of the unordered breaks of numbers to three addends. It turns out that these two different combinatorial problems are related.

Two theorems from the combinatorics and the geometry are used in the proofs and deriving of the formulas .

Theorem 1. For the number of the breaks of the natural number n to three addends is valid

$$p_3(n)=\begin{cases}\dfrac{n^2}{12} & n\equiv 0\ (\mathrm{mod}\,6)\\[2mm] \dfrac{n^2}{12}-\dfrac{1}{12} & n\equiv 1\ (\mathrm{mod}\,6)\\[2mm] \dfrac{n^2}{12}-\dfrac{1}{3} & n\equiv 2\ (\mathrm{mod}\,6)\\[2mm] \dfrac{n^2}{12}+\dfrac{1}{4} & n\equiv 3\ (\mathrm{mod}\,6)\\[2mm] \dfrac{n^2}{12}-\dfrac{1}{3} & n\equiv 4\ (\mathrm{mod}\,6)\\[2mm] \dfrac{n^2}{12}-\dfrac{1}{12} & n\equiv 5\,(\mathrm{mod}\,6)\end{cases}$$

Theorem 2 /Ceva`s theorem by sinuses/. Let ABC be a triangle and the points A_1, B_1 and C_1 lie respectively on the sides BC, CA and AB and they are different from the vertexes of the triangle. Necessary and sufficient condition the lines AA_1, BB_1 and CC_1 to intersect at one point is $\frac{\sin\alpha_2}{\sin\alpha_1}.\frac{\sin\beta_2}{\sin\beta_1}.\frac{\sin\delta_2}{\sin\delta_1}=1.$

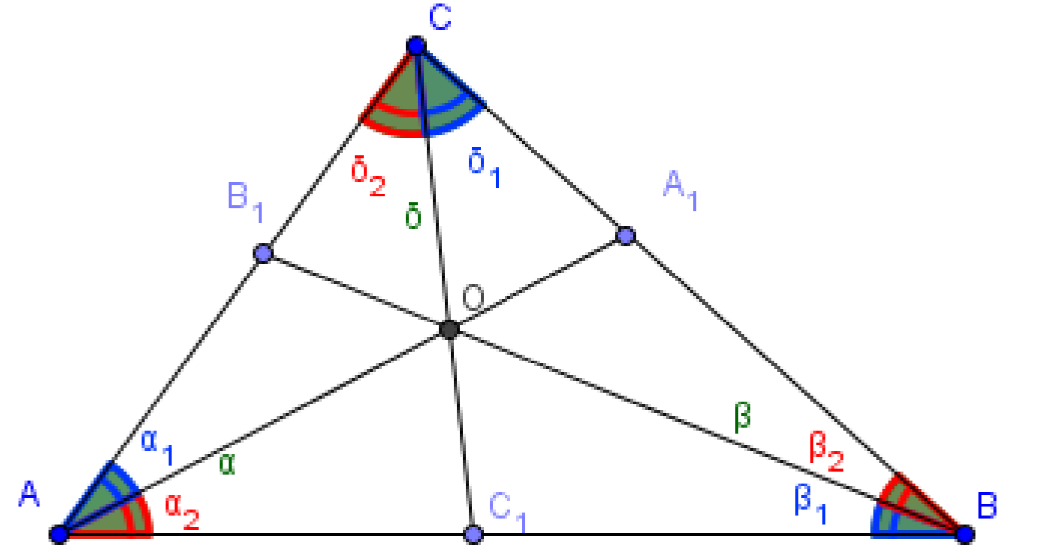

Figure 1.1.

On the basis of the Ceva`s theorem we define some special coordinates of inner points in a circumference.

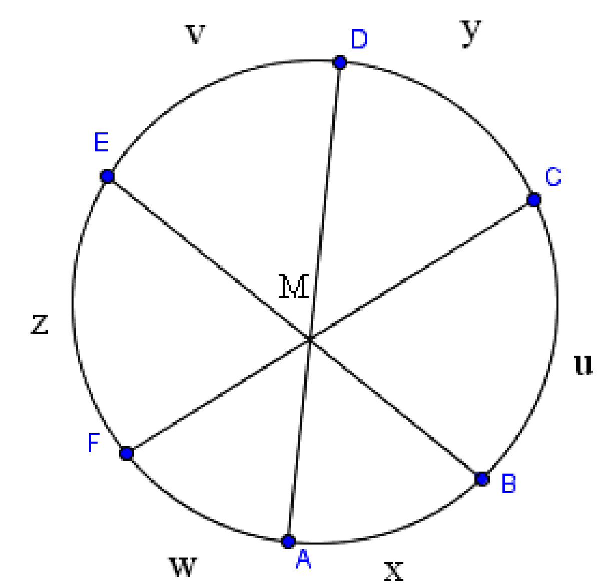

Figure 2.1.

Definition. Let`s consider Figure 2.1. Three chords intersect at point M . Let the arcs on which the circumference separates by the points A, B, C, D, E and F are with measures $\left[(x, y, z);(u, v, w)\right]$ as shown in Fig. 2.1. In this case the ordered sextuple $\left[(x, y, z);(u, v, w)\right]$ will be called coordinates of point M intersection of three chords, respectively three diagonals of the hexagon $ABCDEF$.

Lemma 1 (Author`s). If $\left[(x, y, z);(u, v, w)\right]$ are coordinates of an intersection of three diagonals of a regular polygon, each permutation of triples (x, y, z) and (u, v, w) gives coordinates of other triple diagonals in the regular polygon.

Proof. Follows directly from Ceva`s theorem by shifting the places of multipliers in the numerator and the places of multipliers in the denominator.

Let`s consider now a regular n-gon by $n = 12k + 2$ and $n = 12k - 2$. In these polygons the intersection points of the diagonals are either of two diagonals or of three diagonals except the center where $\frac{n}{2}$ diagonals intersect. In order to find the total number of the intersection points of the diagonals inside the polygon it will be necessary first to determine how many are the points where three diagonals intersect so that we can derive the formula.

Theorem 3 (Author`s). It is given a regular n-gon, where $n = 12k \pm 2$. Then the number of the intersection points of its diagonals inside it is $S_n = \binom{n}{4} - \frac{5n^3 - 45n^2 + 70n - 24}{24}$.

Proof.

A) Let $n = 12k + 2$.

Let`s consider first the cases with the intersection points of a regular 14-gon - it is from the type $n = 12k + 2$.

Figure 2.2 shows a regular 14-gon. Let`s consider the intersection points of the diagonals of a sector A_1OA_2, without one of the limiting radii. The intersection points of three diagonals are K, L, M, N, P, Q, R and S. Their coordinates are

$$K\left[(2,2,3);(2,3,2)\right],\ L\left[(1,2,4);(4,2,1)\right],\ M\left[(2,4,1);(1,4,2)\right],$$

$$N\left[(1,3,3);(3,3,1)\right],\ P\left[(1,4,2);(2,4,1)\right],\ Q\left[(1,5,1);(1,5,1)\right],$$

$$R\left[(1,4,2);(1,4,2)\right] \text{ and } S\left[(1,2,4);(2,4,1)\right].$$

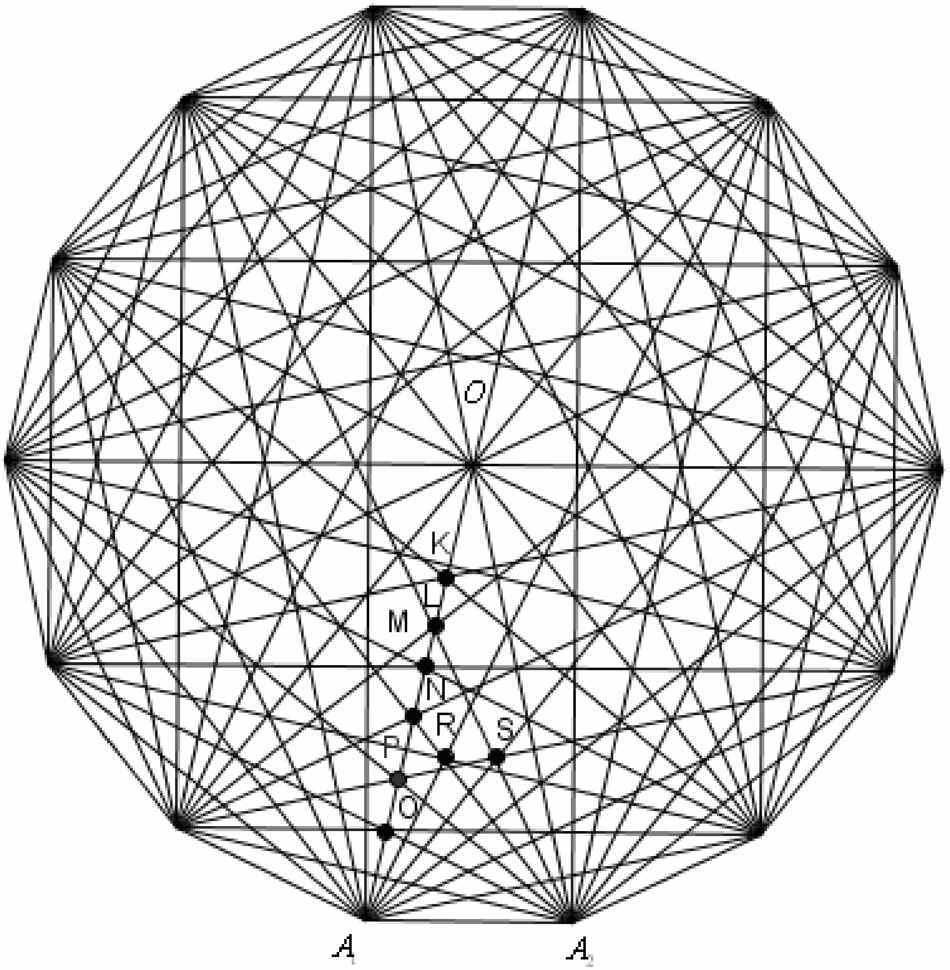

Figure 2.2. Regular 14-gon – the intersection points of the diagonals from one sector A_1OA_2, but without one of the limiting radii.

Here the numbers indicate what part from 2π is the measure of the relevant arc. We notice that the ordered sextuples are composed of ordered triples, each of which is a break of the number 7 to three addends. All those triples are presented $(5,1,1)$, $(4,2,1)$, $(3,3,1)$ and $(3,2,2)$. Their permutations are in the sextuple coordinates as in one sextuple there are triple permutations only of one break of the number 7 to 3 addends.

Let`s consider first one of the vertexes of a regular 14-gon - the vertex A (Figure 2.3). It turns out that all the „triple" intersection points on the diagonals of this vertex have coordinates that contain all the permutations of all breaks of number 7 to 3 addends. (For convenience, the intersections points of three diagonals in a regular polygon will sometimes be called "triple").

Let`s consider now a regular n-gon, where $n = 12k + 2$. We will also use the following statement.

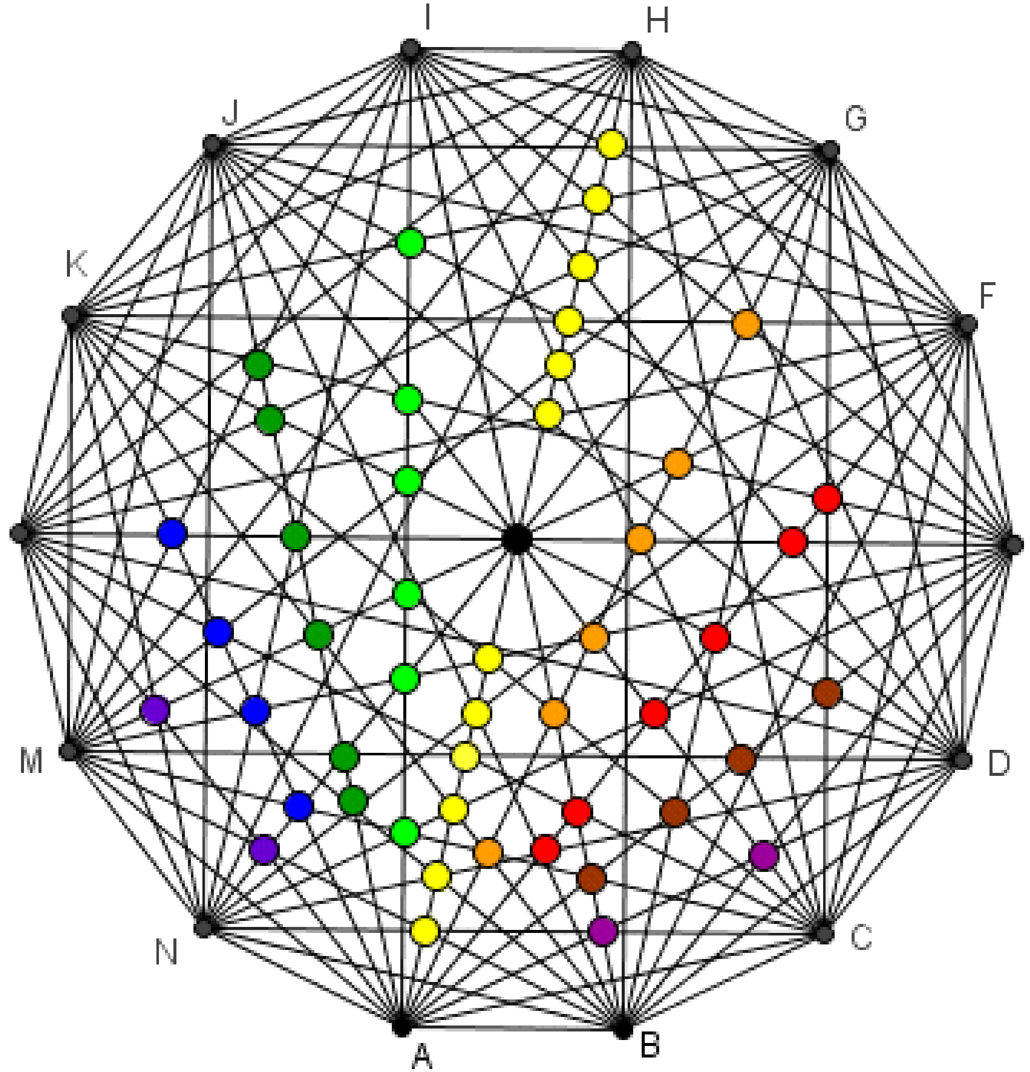

Figure 2.3. All „triple” intersection points on diagonals, which „come out” from the vertex A of a regular 14-gon.

**Lemma 2 (Author`s).** Let it is given a regular $n$-gon, where $n = 12k + 2$ with vertexes $A_1, A_2, ..., A_{12k+2}$. Let`s fix the diagonal $A_1 A_{6k+2}$ which passes through the center O and all triples diagonals through the center, one of which is $A_1 A_{6k+2}$. Then the center has different sextuple coordinates, in which all unordered breaks of the number $6k + 1$ to three addends are presented.

Proof. Let`s consider the diagonals $A_1 A_{6k+2}$, $A_2 A_{6k+3}$ and $A_3 A_{6k+4}$. They define sextuples coordinates of the center $\left[\left(1, 6k-1, 1\right); \left(1, 1, 6k-1\right)\right]$. By selecting another third diagonal, instead of $A_3 A_{6k+4}$, it will present all possible breaks with the number 1 and etc.

Let`s apply the lemma, as first consider all the „triple” points on all diagonals from the vertex A_1 of M_{12k+2} (With M_n we will note a regular n-gon). Their coordinates are ordered sextuples $\left[(x,y,z);(u,v,w)\right]$, in which all permutations of all unordered breaks of the number $6k+1$ to three addends are presented.

Let`s consider a regular 14-gon (Figure 2.5). Compared to starting point A, coordinates of the point I are $\left[(1,5,1);(1,5,1)\right]$ and the coordinates of the point J are $\left[(1,4,2);(4,2,1)\right]$.

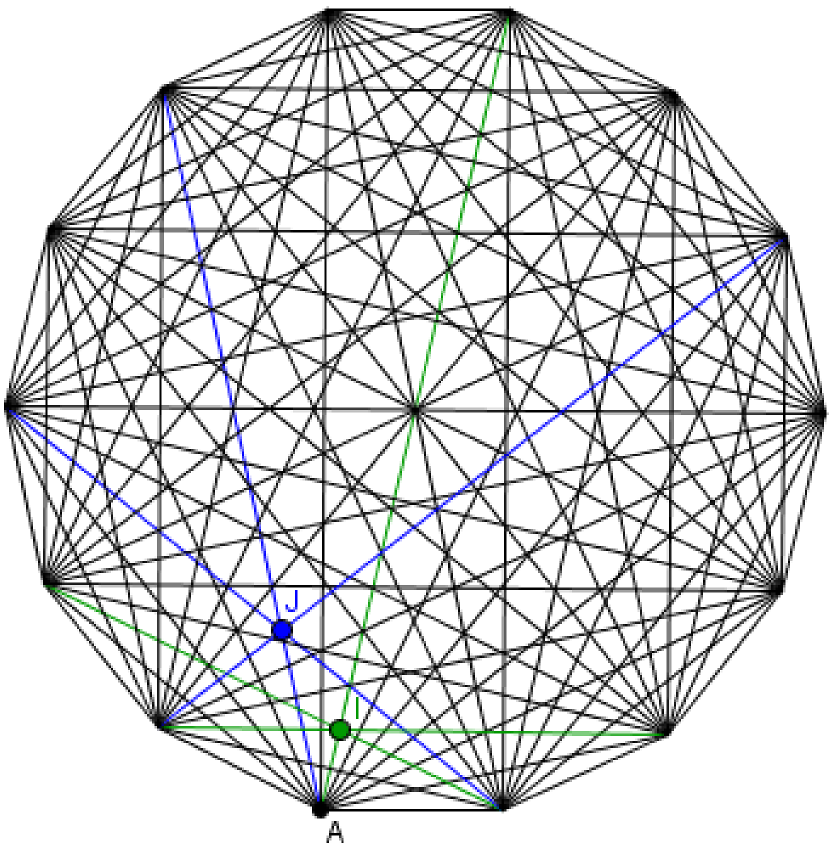

Figure 2.5. A regular 14-gon - coordinates of the point I are $\left[(1,5,1);(1,5,1)\right]$ and the coordinates of the point J are $\left[(1,4,2);(4,2,1)\right]$.

In order to find the intersection points of the diagonals of M_{12k+2} we have to find the number of the “triple” points on the diagonals from one vertex, multiply by $12k+2$ and divide by 6, because each “triple” point will be counted 6 times. After this we have to subtract the derived from the maximum possible number of intersection points of the diagonals inside the $12k+2$-gon.

By $n=12k+2$ the maximum number of diagonals intersecting at one point is 3. The total number of intersection points of inscribed polygon if three diagonals do not intersect at one point is $\binom{n}{4}$.

By $n=12k+2$ the coordinates of our "triple" points are actually made of permutations of breaks of unordered triples of the number $6k+1$.

When there are two equal numbers at one triple, then the sextuples of permutations of this triple are 3^2. This happens as many times as the number of odd numbers less than $6k+1$. The number of odd numbers from 1 to $6k+1$ is $3k$.

When we subtract $3k$ from the total number of breaks, which is $P_3(6k+1)=\frac{(6k+1)^2-1}{2}$, we get the number of breaks in which ordered triples can form 6^2 sextuples.

To get the desired number of „triple" points we need to subtract $3k.3^2+\left(\frac{(6k+1)^2-1}{2}-3k\right).6^2$ from the total number of permutations with the center, which is exactly $\binom{6k}{2}$ as the diagonal pairs, excluding those in point A_1.

Therefore the number of „triple" points through which passes a diagonal with an end point A_1 is $Q_{12k+2} = 3k.3^2\left(\frac{(6k+1)^2-1}{12}-3k\right).6^2-\binom{6k}{2}$. Now from the relation $n=12k+2$ we find $3k=\frac{n-2}{4}$, $6k+1=\frac{n}{2}$, $6k=\frac{n-2}{2}$.

We get $Q_n = \frac{n-2}{4}.3^2+\left(\frac{\left(\frac{n}{2}\right)^2-1}{12}-\frac{n-2}{4}\right).6^2-\binom{\frac{n-2}{2}}{2}=\frac{(n-2)(5n-38)}{8}$.

To find the total number of „triple” points we multiply by n for each vertex and divide by 6, because every point is counted six times. We get the total number of „triple” points

$$T_n = \frac{n}{6} Q_n = \frac{n}{6} . \frac{(n-2)(5n-38)}{8} = \frac{n(n-2)(5n-38)}{48} .$$

To find the total number of intersection points of diagonals in M_{12k+2} we must consider the following:

- Each intersection of three diagonals is a merger of three intersections of two diagonals.

- The center of the regular polygon M_{12k+2} is a merger of the intersection points of pairs of diagonals, which number is

$$\binom{6k+1}{2} = \binom{\frac{n}{2}}{2} = \frac{n(n-2)}{8} ,$$

which are counted for one point.

Therefore the total number of intersection points will be $S_n = \binom{n}{4} - 2T_n - \frac{n(n-2)}{8} + 1$. We calculate

$$2T_n + \frac{n(n-2)}{8} - 1 = \frac{n(n-2)(5n-38)}{24} + \frac{n(n-2)}{8} - 1 =$$

$$= \frac{5n^3 - 45n^2 + 70n - 24}{24}.$$

Therefore $S_n = \binom{n}{4} - \frac{5n^3 - 45n^2 + 70n - 24}{24}$.

B) **Let** $n = 12k - 2$.

Let`s consider a regular 10-gon (Figure 2.6). Compared to starting point A, coordinates of the point K are $\left[(1,2,2);(2,2,1)\right]$ and the coordinates of the point L are $\left[(2,2,1);(2,2,1)\right]$.

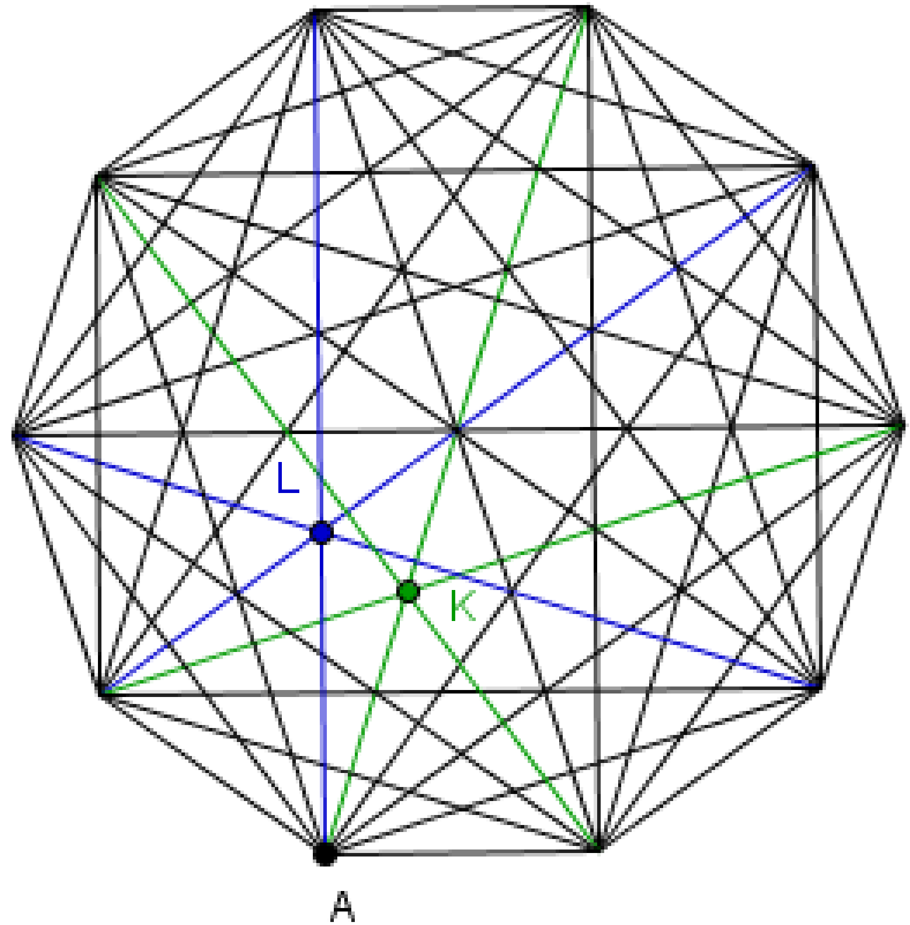

Figure 2.6. A regular 10-gon - coordinates of the point K are $\left[(1,2,2);(2,2,1)\right]$ and the coordinates of the point L are $\left[(2,2,1);(2,2,1)\right]$.

**Lemma 3 (Author`s).** Let it is given a regular $n$-gon, where $n=12k-2$, with vertexes $A_1, A_2, ..., A_{12k-2}$. Let`s fix the diagonal A_1A_{6k} which passes through the center and all triple diagonals through the center, one of which is A_1A_{6k}. Then the center has different sextuple coordinates, in which all unordered breaks of the number $6k-1$ to three addends are presented.

Proof. Analogous to Lemma 2.

Analogous to the previous case, in order to find the number of the intersection points of the diagonals of M_{12k-2}, we need to find the number of the"triple" points on the diagonals from 1 vertex, to multiply by $12k-2$ and divide by 6, because every "triple" points will be counted 6 times.

When from the total number of breaks, which is $P_3(6k-1)=\dfrac{(6k-1)^2-1}{2}$, we subtract $3k-1$, we get the number of breaks, in which ordered triples can form 6^2 sextuples.

To obtain the desired number of „triple" points we need to subtract $\left(3k-1\right).3^2+\left(\frac{\left(6k-1\right)^2-1}{2}-\left(3k-1\right)\right).6^2$ from the total number of permutations with the center, which is exactly $\binom{6k-2}{2}$ as the diagonal pairs, excluding those in point A_1.

Therefore the number of the "triple" points, through which passes a diagonal with an end point A_1 is $Q_{12k-2}=\left(3k-1\right).3^2\left(\frac{\left(6k-1\right)^2-1}{12}-\left(3k-1\right)\right).6^2-\binom{6k-2}{2}$. Now from the relation $n=12k-2$ we find $3k-1=\frac{n-2}{4}$, $6k-1=\frac{n}{2}$, $6k-2=\frac{n-2}{2}$. We get

$$Q_n=\frac{n-2}{4}.3^2+\left(\frac{\left(\frac{n}{2}\right)^2-1}{12}-\frac{n-2}{4}\right).6^2-\binom{\frac{n-2}{2}}{2}=\frac{\left(n-2\right)\left(5n-38\right)}{8}.$$

To find the total number of „triple” points we multiply by n for each vertex and divide by 6, because every point is counted six times. We get the total number of „triple” points

$$T_n = \frac{n}{6} Q_n = \frac{n}{6} . \frac{(n-2)(5n-38)}{8} = \frac{n(n-2)(5n-38)}{48} .$$

To find the total number of intersections of diagonals we must consider the following:

- Each intersection of three diagonals is a merger of three intersections of two diagonals.
- The center of the regular polygon M_{12k-2} is a merger of the intersection points of pairs of diagonals, which number is

$$\binom{6k-1}{2} = \binom{\frac{n}{2}}{2} = \frac{n(n-2)}{8} ,$$

which are counted for one point.

Therefore the total number of the intersection points of the diagonals of M_{12k-2} inside it will be $S_n = \binom{n}{4} - 2T_n - \frac{n(n-2)}{8} + 1$. We calculate

$$2T_n + \frac{n(n-2)}{8} - 1 = = \frac{5n^3 - 45n^2 + 70n - 24}{24}$$

Therefore $S_n = \binom{n}{4} - \frac{5n^3 - 45n^2 + 70n - 24}{24}$.

Note 1. The formulas for a regular $12k+2$-gon and a regular $12k-2$-gon coincide, because this is a result of the coincidence of the respective formulas for the number of breaks.

Note 2. The obtained formula for a regular $12k+2$-gon and a regular $12k-2$-gon coincides with the result, obtained by other means and described in Bjorn Poonen and Michael Rubinstein`s article [1].

Let`s consider now a regular n-gon by $n = 12k + 4$ and $n = 12k - 4$. In these polygons the intersection points of the diagonals are also either of two diagonals or of three diagonals except the center where intersect $\frac{n}{2}$ diagonals. In order to find the total number of the intersection points it is also necessary to determine how many are the points where three diagonals intersect.

Theorem 4 (Author`s). It is given a regular n-gon, where $n = 12k \pm 4$. Then the number of the intersection points of its diagonals inside it is $S_n = \binom{n}{4} - \frac{5n^3 - 45n^2 + 70n - 24}{24} - \frac{3n}{2}$.

Proof. Analogous to theorem 4 we use the following formula for the number of the breaks $p_3\left(6k \pm 2\right) = \frac{\left(6k \pm 2\right)^2 - 4}{12}$.

A) By $n=12k+4$ we respectively have $p_3\left(6k+2\right)=\dfrac{\left(6k+2\right)^2-4}{12}$. The number of the „triple" points, through which a diagonal with an end point A_1 passes is $Q_{12k+4}=3k.3^2+\left(\dfrac{\left(6k+2\right)^2-4}{12}-3k\right).6^2-\dbinom{6k+1}{2}$, from where

$$Q_n=\frac{n-4}{4}.3^2+\left(\frac{\left(\frac{n}{2}\right)^2-4}{12}-\frac{n-4}{4}\right).6^2-\binom{\frac{n-2}{2}}{2}=\frac{\left(n-4\right)\left(5n-28\right)}{8},$$

$T_n=\dfrac{n}{6}Q_n=\dfrac{n}{6}.\dfrac{\left(n-4\right)\left(5n-28\right)}{8}=\dfrac{n\left(n-4\right)\left(5n-28\right)}{48}$ and in the end

$$2T_n+\frac{n\left(n-2\right)}{8}-1==\frac{5n^3-45n^2+70n-24}{24}+\frac{3n}{2}.$$

Therefore $S_n=\dbinom{n}{4}-\dfrac{5n^3-45n^2+70n-24}{24}-\dfrac{3n}{2}$.

B) By $n=12k-4$ we have $p_3\left(6k-2\right)=\dfrac{\left(6k-2\right)^2-4}{12}$,

$$Q_{12k-4}=\left(3k-2\right).3^2+\left(\frac{\left(6k-2\right)^2-4}{12}-\left(3k-2\right)\right).6^2-\binom{6k-3}{2},$$

$$Q_n=\frac{n-4}{4}.3^2+\left(\frac{\left(\frac{n}{2}\right)^2-4}{12}-\frac{n-4}{4}\right).6^2-\binom{\frac{n-2}{2}}{2}=\frac{\left(n-4\right)\left(5n-28\right)}{8}.$$

$$T_n=\frac{n}{6}Q_n=\frac{n}{6}.\frac{\left(n-4\right)\left(5n-28\right)}{8}=\frac{n\left(n-4\right)\left(5n-28\right)}{48}.$$

$$2T_n+\frac{n\left(n-2\right)}{8}-1=\frac{n\left(n-4\right)\left(5n-28\right)}{24}+\frac{3n\left(n-2\right)}{24}-\frac{24}{24}=$$

$$=\frac{5n^3-45n^2+70n-24}{24}+\frac{3n}{2}.$$

Therefore $S_n=\dbinom{n}{4}-\dfrac{5n^3-45n^2+70n-24}{24}-\dfrac{3n}{2}$.

Note 1. The formulas for a regular $12k+4$-gon and a regular $12k-4$-gon coincide, because this is a result of the coincidence of the respective formulas for the number of breaks.

Note 2. The obtained formula for a regular $12k+4$-gon and a regular $12k-4$-gon coincides with the result, obtained by other means and described in Bjorn Poonen and Michael Rubinstein`s article [1].

Thank you for your attention!