Solving equations from the 16th to the 18th century

By Constantine Polyzois and Christos Bitsanis
Varvakeion school of Athens
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Introduction

Galois, whose the bicentennial was celebrated the last year, during his short life, built one of the most fruitful mathematical theories and areas of application wider than we know today. As for the theory of equations, it is not possible to measure the real contribution of Galois without having read what had been done before him.

The classical problem of solving an n^{th} degree polynomial equation

An $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ (1)

has substantially influenced the development of mathematics throughout the centuries and has still several important applications to the theory and practice of present-day computing.

The aims of this paper is to show same aspects of solving algebraic equations of degree 3 beginning in the sixteenth century with the Italian mathematicians and ends in the eighteenth century with the work of Lagrange.

Khayyam's resolution

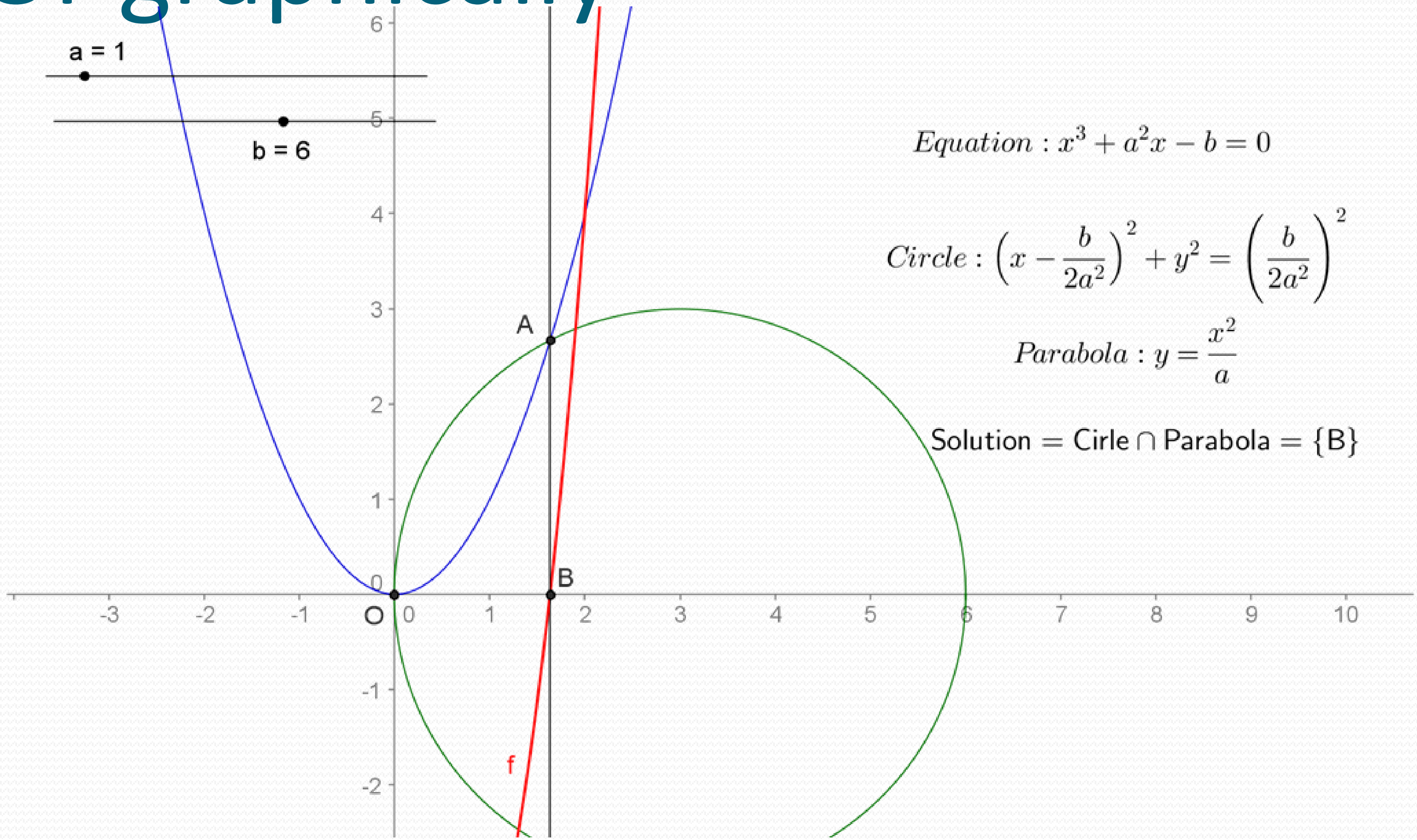
Let, f(x) = $x^3 + a^2x - b = 0$ (1)

Equation (1) transformed into $\begin{cases} \left(x - \frac{b}{2a^2}\right)^2 + y^2 = \left(\frac{b}{2a^2}\right)^2 \\ y = \frac{x^2}{a} \end{cases}$

Let: $x^3 + 1.5^2 x - 7.1 = 0$

is equivalent to $\begin{cases} x^2 + y^2 - 3.16x = 0 \\ y = 0.67x^2 \end{cases}$

Or graphically



$$Equation : x^3 + a^2x - b = 0$$

$$Circle : \left(x - \frac{b}{2a^2}\right)^2 + y^2 = \left(\frac{b}{2a^2}\right)^2$$

$$Parabola : y = \frac{x^2}{a}$$

$$\text{Solution} = \text{Cirle} \cap \text{Parabola} = \{\text{B}\}$$

Cardano’s method

Let,

$$x^3 + ax^2 + bx + c = 0$$

$$x = y - \frac{a}{3}$$

$$x^3 = y^3 - ay^2 + \ldots \quad (1)$$

$$ax^2 = ay^2 + \ldots. \quad (2)$$

$$x^3 + ax^2 + bx + c \overset{(1)}{\underset{(2)}{=}} y^3 - y\left(-\frac{a}{3} + b\right) + \frac{2}{27}a^3 - \frac{1}{3}ba + c$$

$$= y^3 + py + q$$

Cardano’s method

$$x^3 + px + q = 0$$

$$x = u + v$$

$$x^3 + px + q \quad = \quad (u+v)^3 + (3uv + p)(u+v)$$

$$\text{Let}: uv = -\frac{p}{3}$$

$$(u+v)^3 + (3uv+p)(u+v) \quad = \quad \begin{cases} u^3 + v^3 = -q^3 \\ u^3 v^3 = -\left(\dfrac{p}{3}\right) \end{cases}$$

Cardano's method

$$(x-u^3)(x-v^3)=\ldots=x^2+qx-\left(\frac{p}{3}\right)^3$$

$$u=\sqrt[3]{\frac{-q+\sqrt{q^2+4\left(\frac{p}{3}\right)^3}}{2}},\quad v=\sqrt[3]{\frac{-q-\sqrt{q^2+4\left(\frac{p}{3}\right)^3}}{2}}$$

$$\text{Or}, x=u+v=\sqrt[3]{\frac{-q+\sqrt{q^2+4\left(\frac{p}{3}\right)^3}}{2}}+\sqrt[3]{\frac{-q-\sqrt{q^2+4\left(\frac{p}{3}\right)^3}}{2}}$$

The symmetrical functions

- **Definition:** One algebraic relation between the variables x, y, z, v,… say that is *symmetrical*, if one permutes the variables of them, then this quantity remain unchanged
- **For example**

  $x^3+bx^2+cx+d=(x-r_1)(x-r_2)(x-r_3)$
- Then the Vieta's formulas is symmetric functions

$$\begin{cases} r_1+r_2+r_3=-b \\ r_1r_2+r_2r_3+r_1r_3=c \\ r_1r_2r_3=-d \end{cases}$$

The Newton's theorem

- **Newton's Theorem:** Every symmetric function of the roots of a polynomial, then it can be expressed in terms of coefficients of this polynomial by sums and products.
- For example

$$r_1^3 + r_2^3 + r_3^3 = -b^3 + 3bc - 3d$$

Lagrange's method

- $x^3 + bx^2 + cx + d = (x - r_1)(x - r_2)(x - r_3) = 0$ (1)

$$\text{Viete:=}\begin{cases} v_1 : r_1 + r_2 + r_3 = -b \\ v_2 : r_1 r_2 + r_2 r_3 + r_1 r_3 = c \\ v_3 : r_1 r_2 r_3 = -d \end{cases}$$

$$\begin{cases} u = r_1 + \rho r_2 + \rho^2 r_3 \\ v = r_1 + \rho^2 r_2 + \rho r_3 \end{cases}$$

Where ρ is a complex number ≠1 such that: $\rho^3=1$, for example: $\rho = \dfrac{-1 + i\sqrt{3}}{2}$

$$\begin{cases} u = r_1 + \rho r_2 + \rho^2 r_3 & v = r_1 + \rho^2 r_2 + \rho r_3 \\ u' = r_2 + \rho r_3 + \rho^2 r_1 & v' = r_3 + \rho^2 r_1 + \rho r_2 \\ u'' = r_3 + \rho r_1 + \rho^2 r_2 & v'' = r_2 + \rho^2 r_3 + \rho r_1 \end{cases}$$

Then: u^3=$u`^3$=$u``^3$ and $v^3$=$v`^3$=$v``^3$.

Lagrange’s method

- uv = $\left(r_1^2 - r_1 \cdot r_2 - r_1 \cdot r_3 - r_2 \cdot r_3 + r_2^2 + r_3^2\right)$
- $u^3 + v^3 = (2r_1 - r_2 - r_3)(2r_2 - r_1 - r_3)(2r_3 - r_2 - r_1)$
- By Newton’s theorem these two quantities can be expressed in terms of coefficients of this polynomial by sums and products. So:

$$\begin{cases} uv = v_1^2 - 3v_2 = b^2 - 3c \\ u^3 + v^3 = 2v_1^3 - 9v_1v_2 + 27v_3 = 2(-b)^3 - 9(-b)c + 27(-d) \end{cases}$$

Lagrange's method

- But, the u^3 and v^3 are the solution of the polynomial

$y^2 - (u^3 + v^3)y + (u^3 v^3)$

- Then, we have the system:

$$\begin{cases} v_1 = r_1 + r_2 + r_3 \\ u = r_1 + \rho r_2 + \rho^2 r_3 \\ v = r_1 + \rho r_3 + \rho^2 r_2 \end{cases} \Leftrightarrow \begin{cases} r1 = \dfrac{v_1 + u + v}{3} \\ r2 = \dfrac{v_1 + \rho^2 u + \rho v}{3} \\ r3 = \dfrac{v_1 + \rho u + \rho^2 v}{3} \end{cases}$$

Example (Cardano's formula)

- Given the polynomial $x^3 + px + q = 0$ (1)
- Then : $$Viete := \begin{cases} v_1 : r_1 + r_2 + r_3 = 0 \\ v_2 : r_1 r_2 + r_2 r_3 + r_1 r_3 = p \\ v_3 : r_1 r_2 r_3 = -q \end{cases}$$
- Lagrange intermediate quantities: $$\begin{cases} u = r_1 + \rho r_2 + \rho^2 r_3 \\ v = r_1 + \rho^2 r_2 + \rho r_3 \end{cases}$$

$$\begin{cases} uv = v_1^2 - 3v_2 = -3p \\ u^3 + v^3 = 2v_1^3 - 9v_1 v_2 + 27v_3 = -27q \end{cases}$$

- Solving the equation:
- $y^2 + 27qy - 27p^3 = 0 \quad \Leftrightarrow u^3, v^3 = -\frac{27}{2} q \pm \frac{3}{2}\sqrt{81 \cdot q^2 + 12 \cdot p^3}$

Example (Cardano's formula)

$$\begin{cases} v_1 = r_1 + r_2 + r_3 \\ u = r_1 + \rho r_2 + \rho^2 r_3 \\ v = r_1 + \rho r_3 + \rho^2 r_2 \end{cases} \Leftrightarrow \begin{cases} r_1 = \dfrac{v_1 + u + v}{3} \\ r_2 = \dfrac{v_1 + \rho^2 u + \rho v}{3} \\ r_3 = \dfrac{v_1 + \rho u + \rho^2 v}{3} \end{cases}$$

$$\Leftrightarrow \begin{cases} r_1 = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} + \sqrt[3]{-\dfrac{q}{2} - \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} \\ r_2 = \rho^2 \sqrt[3]{-\dfrac{q}{2} + \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} + \rho \sqrt[3]{-\dfrac{q}{2} - \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} \\ r_3 = \rho \sqrt[3]{-\dfrac{q}{2} + \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} + \rho^2 \sqrt[3]{-\dfrac{q}{2} - \sqrt{\left(\dfrac{p}{3}\right)^3 + \left(\dfrac{q}{2}\right)^2}} \end{cases} \quad \textbf{(S)}$$

Example (Cardano's formula)

- take the equation $x^3 - 8x - 8 = 0$

$$x^3 - 8x - 8 = 0 \Leftrightarrow (x+2)\left[x-(1+\sqrt{5})\right]\left[x-(1-\sqrt{5})\right] \Leftrightarrow r_1 = -2,\ \ r_2 = 1+\sqrt{5},\ \ r_3 = 1-\sqrt{5}$$

- By the previous formulas (S), if $\rho = \dfrac{-1+i\sqrt{3}}{2}$, $i = \sqrt{-1}$

$$\begin{cases} r_1 = \sqrt[3]{4+\frac{4i}{9}\sqrt{15}} + \sqrt[3]{4-\frac{4i}{9}\sqrt{15}} \\ r_2 = \rho^2 \cdot \sqrt[3]{4+\frac{4i}{9}\sqrt{15}} + \rho \cdot \sqrt[3]{4-\frac{4i}{9}\sqrt{15}} \\ r_3 = \rho \cdot \sqrt[3]{4+\frac{4i}{9}\sqrt{15}} + \rho^2 \cdot \sqrt[3]{4-\frac{4i}{9}\sqrt{15}} \end{cases} \underset{\text{Maple}}{\overset{\text{simplify}}{\Leftrightarrow}} \begin{cases} r_1 = 1+\sqrt{5} \\ r_2 = 1-\sqrt{5} \\ r_3 = -2 \end{cases}$$

The Galois role

- Lagrange did not stop there and even his method of solving equations of degree great than 4 but in very special cases. Abel and Galois, relying both on the work of Lagrange, will eventually show independently of one another that the equation of degree 5 is not solvable by radicals.
- Galois goes further and completely the answer about the solvability of equations by giving a necessary and sufficient condition for an equation is solvable by radicals. The structure of all the permutations of n roots, in Galois theory, is consider as a mathematical object, this structure takes the name of Galois group. Each property of an equation translates into a property of the corresponding group Galois and vice versa.

Conclusion

We would like to thank for his help, our professor of mathematics Mr. Zenon Lygatsikas.

Bibliography

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Thank you for your attention !!!