Incommensurability and the Pentagon

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- ✓ Mathematics of 7th Book of Euclid's Elements is ascribed to Pythagoreans
- ✓ In this Book it is described a **method for finding the greatest common divisor** of two or more numbers based in a process called **anthyphairesis**.
- ✓ This process consists of successive mutual subtractions (anthyphairesis) and is described by Euclid in propositions 1 and 2 of the 7th Book of the Elements
- ✓ We will describe this method using an example. We want to find the greatest common divisor of numbers 368 and 136

1st stage: $368 = 136 \cdot 2 + 96$ and $96<136$

2nd stage: $136 = 96 \cdot 1 + 40$ and $40<96$

3rd stage: $96 = 40 \cdot 2 + 16$ and $16<40$

4th stage: $40= 16 \cdot 2 + 8$ and $8<16$

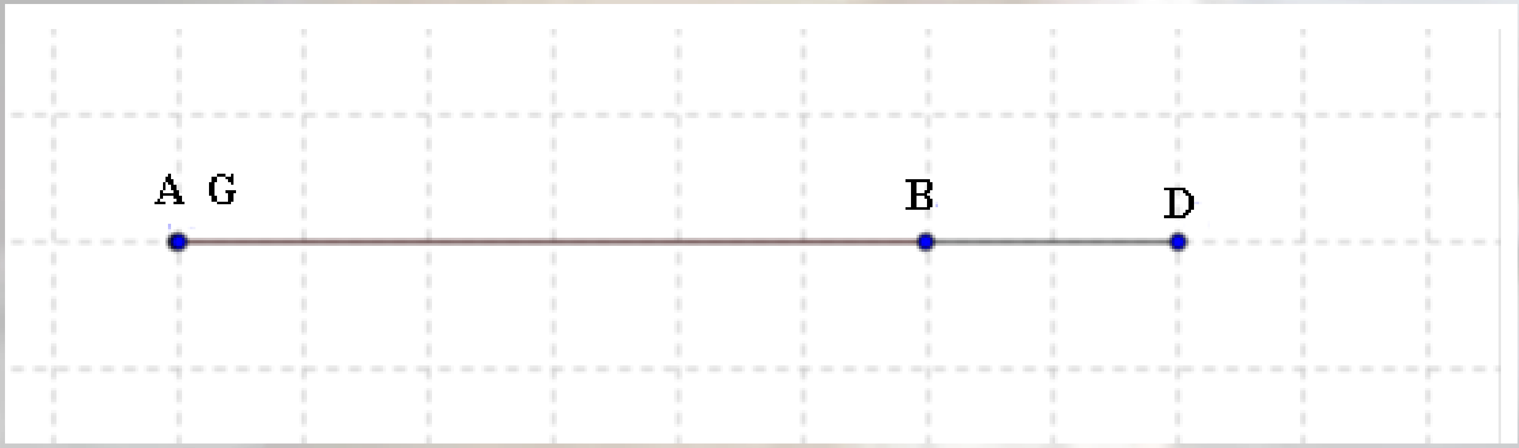
5th stage: $16 = 8 \cdot 2 + 0$ and $0<8$

- ✓ The greatest common divisor is the last remainder which is not zero, in the other words, it is the number **8**.

- ✓ The process of anthyphairesis is used also in the 10th Book of the Elements for finding the greatest common measure of two magnitudes.
- ✓ We know that two segments are not always commensurable. This was known in Euclid's time but was not known by Pythagoreans, till they discovered the existence of incommensurable magnitudes.

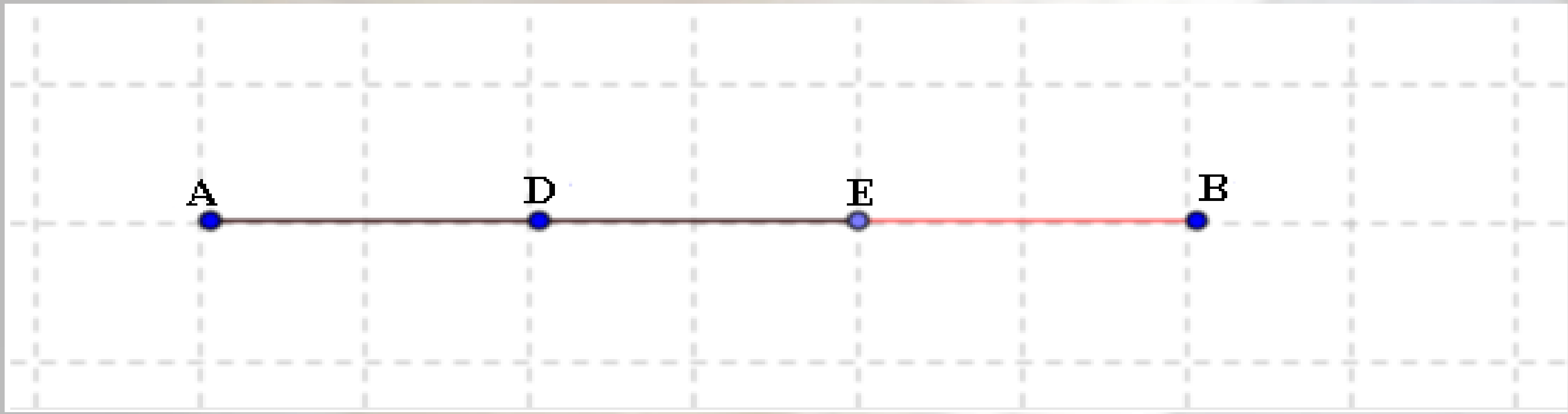
We will describe this method using an example. We want to find the greatest common measure of two segments AB and GD.

<u>1st stage:</u> we 'divide' the two parts:



We can see that the remainder of AB and GD is the segment BD.

<u>2st stage:</u> We repeat the process with the segments AB and BD



We observe that BD measures the segment AB and no remainder is left.

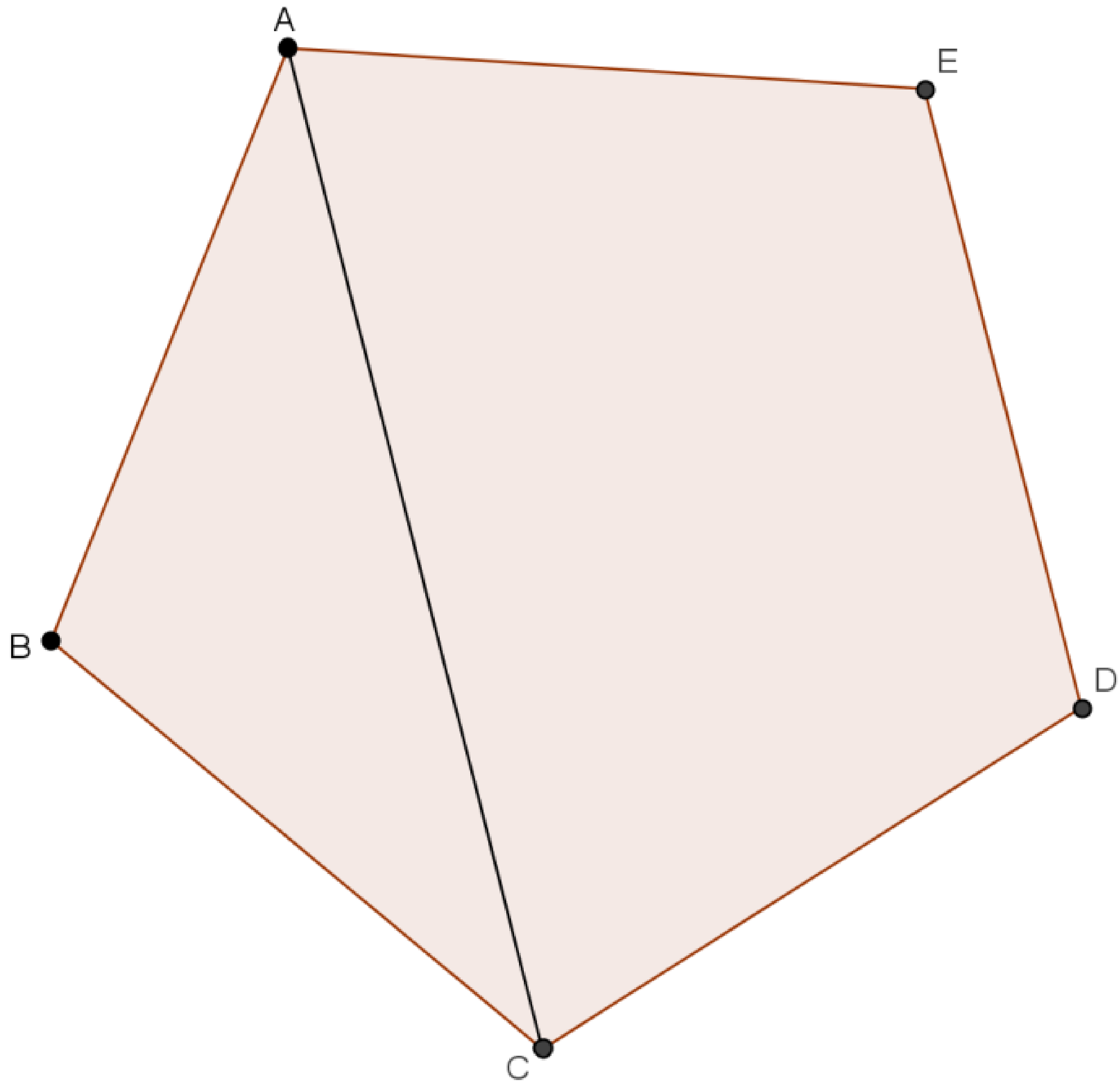
So the process is finished. The last not zero remainder, the segment BD is the greatest common part which can measure both segments AB and GD.

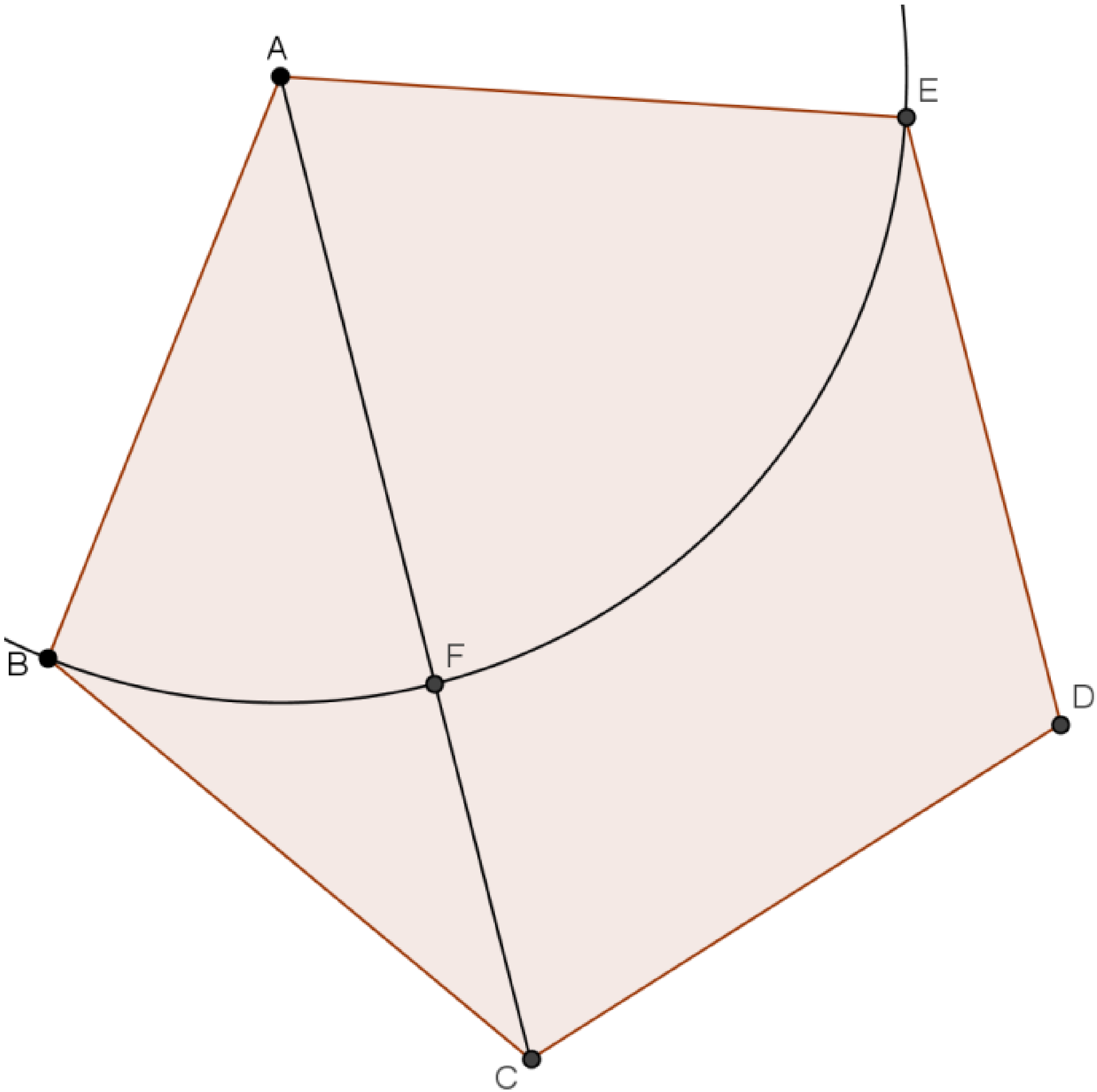
- ✓ If we do not know that there are incommensurable magnitudes, we expect that every two segments should have a common measure, that is that in some stage we will find no remainder and the process of anthyphairesis is always finite.
- ✓ So, if we choose two segments which are really incommensurable, but we believe that they are commensurable, what will happen? We can suppose that we will follow the stages of mutual subtraction again and again many times, but in every stage we will believe that anthyphairesis is just a very long one.

- ✓ Is there an example of two incommensurable segments whose antyphairesis is obviously an infinite one?

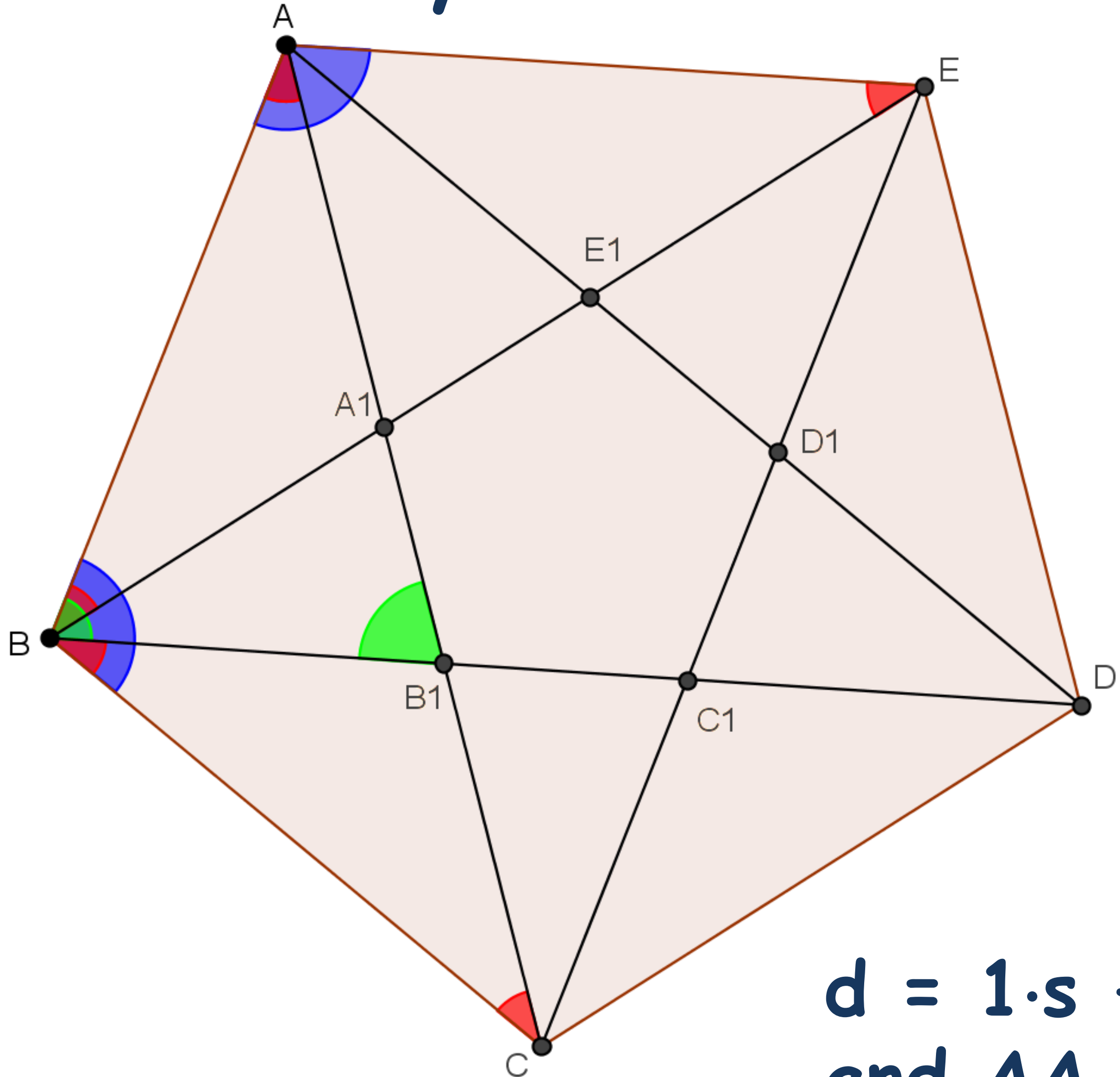
- ✓ We will try to find the greatest common measure of the diameter and the side of a regular pentagon, and try to show that their anthyphairesis is **obviously** an infinite one.

The anthyphairesis of the diameter and side of a regular pentagon

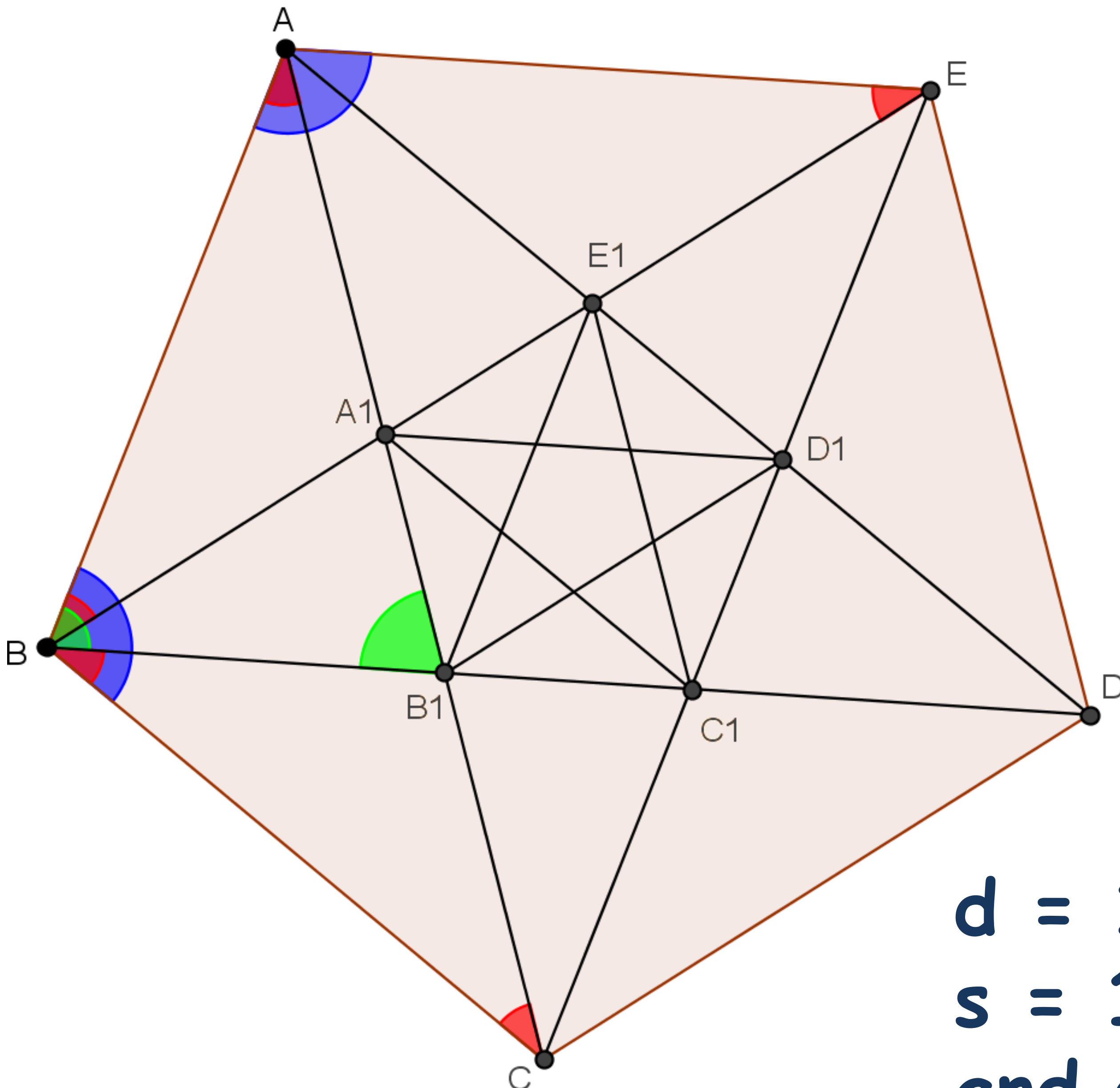



A
E
F
B
D
C

Auxiliary constructions

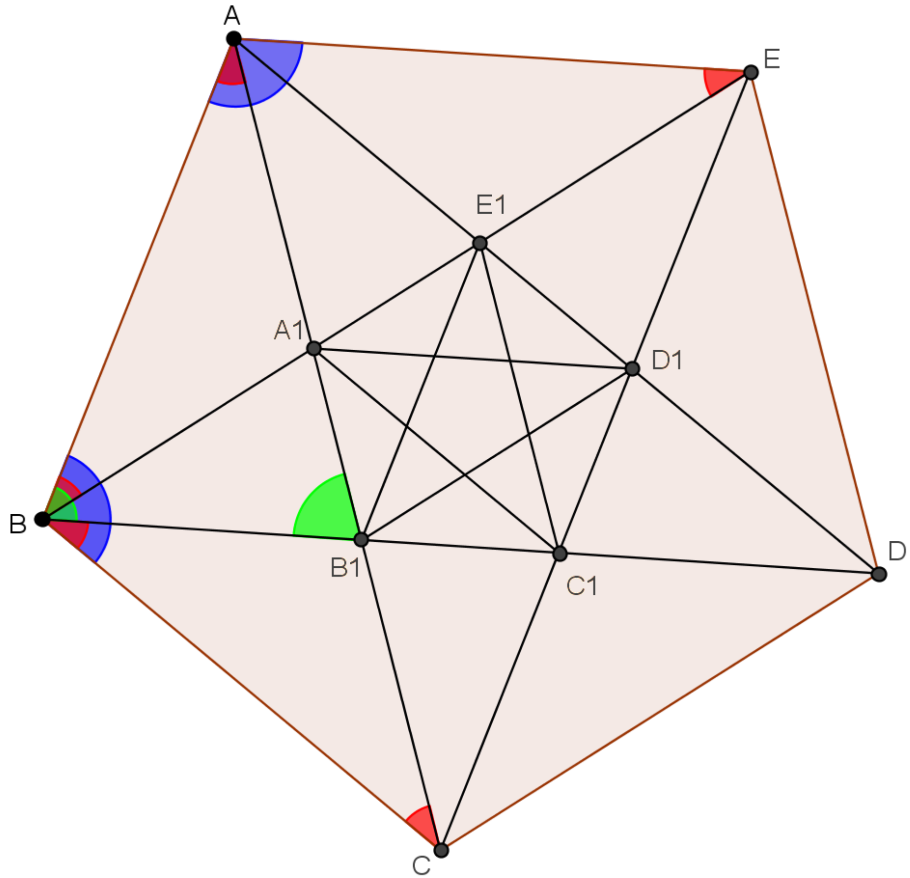


$d = 1 \cdot s + AA_1$
and $AA_1 < s$

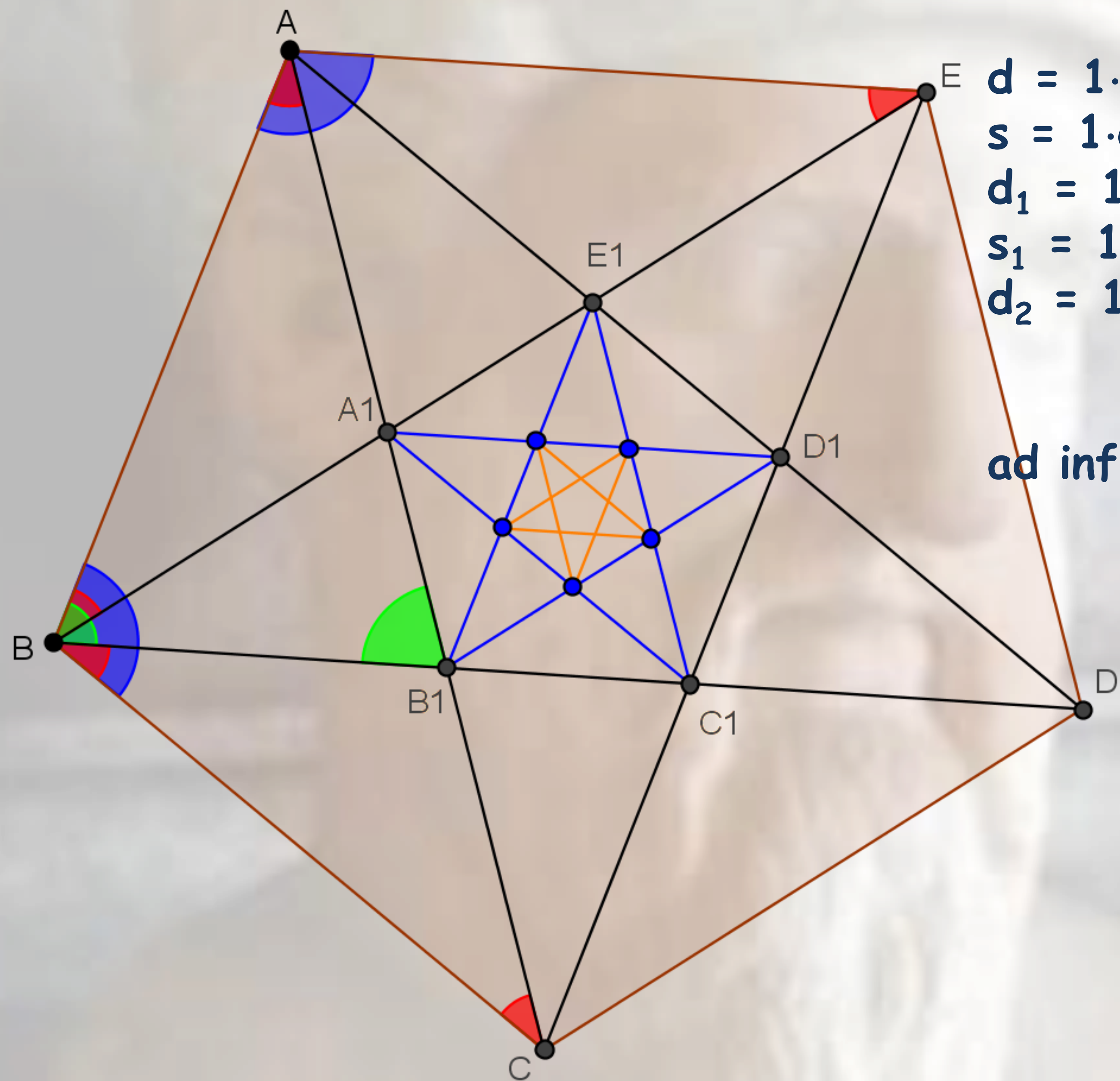


$$d = 1 \cdot s + d_1$$
$$s = 1 \cdot d_1 + s_1$$
$$\text{and } d_1 < s, \; s_1 < d_1$$

- ✓ **For the third stage of anthyphairesis** we have to subtract from A_1C_1 (the diameter of a regular pentagon) A_1B_1 (the side of the same regular pentagon).



- ✓ It is obvious that we have again the same problem as that we had to solve (the anthyphairesis of the diameter and the side of a regular pentagon) and that the periodicity that appears leads us to an infinite process.
- ✓ If we denote the side and the diameter of the n^{th} pentagon, shaped from diameters of $(n-1)^{th}$, as s_n and d_n, then we have an infinite antyphairesis.



$$d = 1 \cdot s + d_1 \qquad s > d_1$$

$$s = 1 \cdot d_1 + s_1 \qquad d_1 > s_1$$

$$d_1 = 1 \cdot s_1 + d_2 \qquad d_2 > s_1$$

$$s_1 = 1 \cdot d_2 + s_2 \qquad d_2 > s_2$$

$$d_2 = 1 \cdot s_2 + d_3 \qquad s_2 > d_3$$

$$\vdots$$

$$\vdots$$

ad infinitum

We can now infer (at least) that something strange happens with the existence of the greater common measure of the diameter and the side of a regular pentagon.

Thank you