FINDING THE ALGEBRAICAL RULE IN GEOMETRIC PATTERNS

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ABSTRACT

In this paper we will deal with the multiple solutions to some problems with geometric patterns. These geometric patterns are sequences of distinct collections of iconic elements such as dots or squares. In these problems there are a sufficient number of terms, from which we can form all the other terms and find the number of elements for any term of the sequence. Our goal is to discover the general rule, to formulate it in algebraic expression and explain it. The problems, which are presented, are of linear and quadratic type. Some of them have historical roots, and are connected with the geometric representations of numbers by the Pythagoreans. As soon as we have solved problems using a variety of different methods, we have created something original. The joy of discovery is a unique experience and we would like to share it with you.

Introduction and our motivations

The word "pattern" refers to the rule which governs a situation or a phenomenon. In particular, according to the rule of geometric pattern, sequences are formed by distinct collections of figurative elements, such as dots or squares. In the problems of patterns a sufficient number of terms, is usually given and we can find the rule from which all terms are formed and predict the number of elements for any term of the sequence.

The patterns are found in arithmetic and geometric sequences problems, as well as in various other real situations. Mathematics is considered the science of patterns and order (Schoenfeld, 1992). The algebraic reasoning consists of different types of generalization of mathematical concepts starting from a set of specific cases.

In addition we can perceive the same pattern differently. This means that it is possible for the same pattern to reach different generalizations (Rivera & Becker, 2008).

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In this study we will examine and justify the solution method to two problems with geometric patterns of quadratic type, concentrating on the algebraic generalizations that we discovered.

Our first contact with geometrical patterns' problems came from the Internet, from the journal of the Hellenic Mathematical Society "*Euclid A"* for the high school students as well as books written for mathematical competitions, such as publications of the AMC. From these sources we have found same amazing problems that we have solved with pleasure.

What encouraged our interest? It started when our maths teacher organized a special program (during the school year 2008-9) on the topic "*Mathematics and Literature*".

The project was organized outside schools hours and student participation was voluntary. We were very keen on this project as it held our attention. During the program we read the book "*The Prince of Mathematics*" (Tent, 2006) which exposed the discoveries of the famous mathematician Carl Friedrich Gauss. We were impressed by the excellent ability of Gauss's calculation concerning the sum 1+2 +3 +...+100=5050, when he was only 7 years old. We also learnt about the gnomons of Pythagoras (right angles) and the calculation of the sum of the consecutive odd numbers. For example: $1+3+5+7=4^2$, $1+3+5+7+9=5^2$ κτλ.

Moreover, in the regular lesson the same teacher taught us the functions of the form y=ax+b and she gave us the problem below:

We consider the following sequence of squares made of toothpicks.

1 2 3

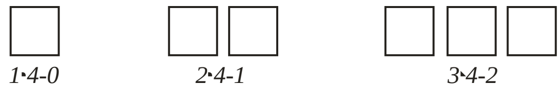
Find a formula for the total number of toothpicks T which are needed to form n squares (the shape with the number n).

Our classmates solved the problem in different methods. Some were right and some were wrong. We found the following solutions:

1+1·3 1+2·3 1+3·3

Therefore the solution is: $T=1+n\times3$.

We also found the solution the solution billow:

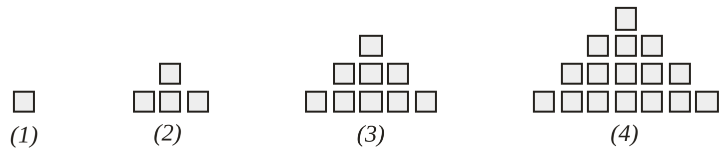


The formula for the nth shape is: $T= n\times4 - (n-1)$

We both felt delighted by the discovery of the solution. It was something like “Eureka of Archimedes!” This whetted our appetite and we solved similar problems, of linear type. Then, we wondered whether there are problems with geometric pattern of quadratic type. So, we found a problem, we formulated a second problem and we solved them. Following a lot of discussion we organized our work in the manner below.

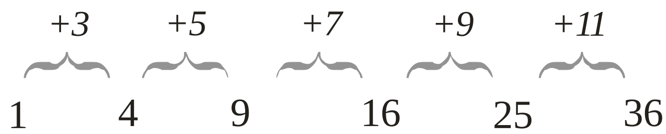
The first problem and its multiple solutions

PROBLEM 1: *We consider the following sequence of squares that don’t cover each other.*



(a) *How many squares are there in shape (5)? Design and explain.*
(b) *How many squares are there in shape (100)? Explain.*
(c) *Find a formula for the total number of squares in the figure (n). Explain how you found your answer?*

Initially, we noticed that the number of squares in the 4 shapes are increased by 3, 5, 7 etc.



We started from 1 and we found the second term by adding 3. So, we found 4 (1+3=4). Then, we added 5 to 4 which is 3 increased by 2 (5-3=2). We found 9 (1+3+5=9). Then we added 7 to 9 which is 3 increased by 2 (7-5=2). We found 16 (1+3+5+7=16). We were able to continue in the same way until we found the correct total number of squares.

The terms of the sequence start from 1 and increase. However, the difference of any two consecutive terms wasn’t stable. As a result, it isn’t an arithmetic series.

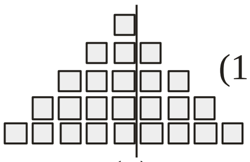
This pattern led us to discover the arithmetic solution. We noticed that “perfect squares” are formed. We have: 1^2, 2^2, 3^2, 4^2, 5^2.

So, we guessed the following rule: n^2. We are sure that each term of the sequence can be derived from this formula. Thus, the 100th shape will contain 100^2 squares.

However, this facility didn't convince us. We wanted to prove it. After much thought we found three more ways of algebraic generalization that we display below.

1. The two terms addition method : (1+2+3+…+n)+(1+2+…+n-1)

In this method, the whole shape consists of two sections. The whole geometrical figure is a composition of two "triangles of squares".



(1+2+3+4+5)+(1+2+3+4)=15+10=25

(5)

From a "sample" of four examples we found the solution for shape (5), we passed to shape (100) and finally we found an algebraic generalization of the solution.

In the same way that we split the 5th shape above, we split the 100th shape into two triangular shapes. We imagined this, because it is impossible to construct it. So we have two big triangles. The first reaches 100 and the second 99: 1+2+3+…+ 100 and 1+2+3+…+ 99. The sum 1+2+3+…+100 is $\frac{100 \cdot 101}{2}$*. At school we had solved the following problem: "There are 7 students. If they greet each other how many handshakes will there be?" As the students are 7, then each one shakes hands with 6 others and so all the handshakes are 7×6, or 42. But the handshakes are counted twice. So, the right number is* $\frac{6 \cdot 7}{2} = 21$ *handshakes. In the same way, we find 1+2+3+…+100=*$\frac{100 \cdot 101}{2} = 5050$*. As a result, all the squares will be: 10.100-100=10.000. We remove 100 because it was counted twice in the middle column. If we put 100 in the place of n we find: 1+2+3+…+n* $= \frac{n(n+1)}{2}$*. Consequently, we have:*

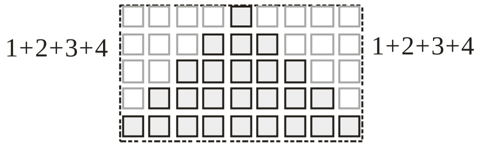
$$(1+2++n)+(1+2+\ldots+n-1)= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n}{2}\left[(n+1)+(n-1)\right] = \frac{n \times 2n}{2} = n^2 .$$

In the previous solution we used the shape and we did algebraic and arithmetic operations. The information we got from observing shape (5), is that this isn't only valid for the 100th shape but also for the nth and to the nth shape.

2. The removal from the rectangle: (2n-1)·n-2·(1+…+n-1)= n^2

The shape results from imagining a large rectangle and removing the two sections located between the rectangle and the original shape. For the 5th square we have:

$$9 \cdot 5 - 2 \cdot (1+2+3+4) = 45 - 2 \cdot 10 = 45 - 20 = 25$$



(5)

The shape (n) will contain: (2n-1)·n-2·(1+2+3+…+n-1) squares. Just in the problem of handshakes for 7 students we found $\frac{6 \cdot 7}{2} = 21$ *handshakes. Generally a triangle of squares will have:* $1+2+3+\ldots+n-1 = \frac{(n-1) \cdot n}{2}$ *squares. Finally, we find:*

$$(2n-1) \cdot n - 2 \cdot (1+2+3+\ldots+n-1) = 2n^2 - n - 2\,\frac{(n-1) \cdot n}{2} = 2n^2 - n - n^2 + n = n^2.$$

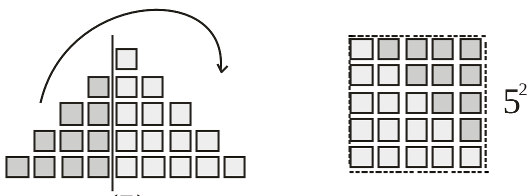
So, the total number of squares will be n^2 *squares.*

In this case it was necessary to imagine a rectangle of length 2n-1 and width n from which two equal sums of the form 1+2+3+...+n-1are removed. The previous solution was achieved by productive use of prior knowledge.

3. The method of reconstruction: $1+3+5+\ldots+2n-1=n^2$

In this way we rebuild the figure.

First of all, we constructed the 5th shape. The base is 9 squares and the height 5 squares. We cut the shape into two "triangles of squares", we moved the left triangle of squares and we placed it above the right triangle of squares. So, we formed a square 5×5 with area 25.



(5)

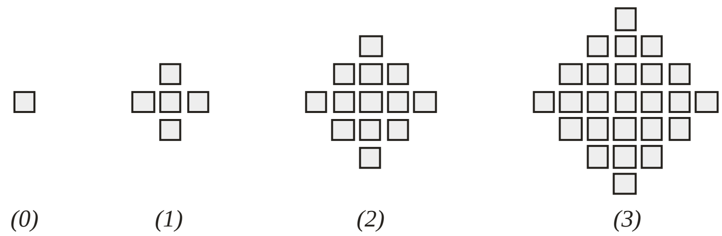
In the square we have two right and isosceles "triangles of squares": 5×5 and 4×4. We obtain a square with side 5. It has 25 squares. The 100^{th} *shape will have a height of 100 , ie 10,000 squares. And finally, the nth shape has n × n squares.*

In our opinion, the previous way surpasses the others because it is very simple, since it doesn't require the calculations of totals such as 1+2+3+... +n or 1+3+5+...+2n-1. To solve it, we only had to look at the shape, we split it and we reunited the two parts in a different way. After the geometric reorganization, it is clear!

To sum up, we recognized the solution, then we went on a description and finally we solved it algebraically in three different ways. The algebraic generalization requires close observation, comparison of the terms of a small "sample" and the discovery of the "connective tissue".

The second problem and the different solution methods

PROBLEM 2: *We consider the following sequence of squares that don't cover each other.*



(a) *How many squares are there in shape (4)? Design and explain.*

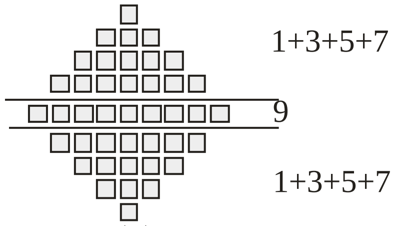
(b) *How many squares are there in shape (100)? Explain.*

(c) *Find a formula for the total number of squares in figure (n). Explain how you found your answer?*

We started solving the problem by testing and verifying. The rule 1+4n was first thought of, because if we put n=0 we find 1 and if we put n=1 then we find 5. But it doesn't satisfy the following term since for n=2 we find 9 and not 13. After that, we thought of the formula 1+6n, but it only works for the first and the third term. The form of induction that is used in the previous rules is called "simple induction" and differs from other more sophisticated forms of induction (Radford, 2008). Initially, we were unable to free our mind from the arithmetical exploration and the "trial and error method" and find a general rule. Soon we abandoned the method of trial and error, which led us to arbitrary rules. After much thought we found 3 ways which are described below:

1. Addition of three terms : $2(1+3+5+\ldots+2n-1)+2n+1=2n^2+2n+1$

According to this way the whole shape is a synthesis of other shapes. The whole figure is cut into two equal "triangles of squares" and a horizontal row of squares between them. As it is shown in the following shape, the 4th figure consists of these parts.



(4)

To find the number of squares it is enough to add the three parts.

Figure (4) contains 2×(1+3+5+7)+9 squares. Figure (100) will consist of 2×(1+3+5+...+197+199)+201 squares. The parenthesis has 100 terms, because it increases two by two. For instance, the sum 1+3+5+7+9+11+13+15+17+19 has 10 numbers and the sum up to 199 has to contain 100 numbers. We have:

$$2\cdot(1+3+5+\ldots+197+199)+201=2\cdot 100^2+201=20.000+201=20.201 \text{ squares.}$$

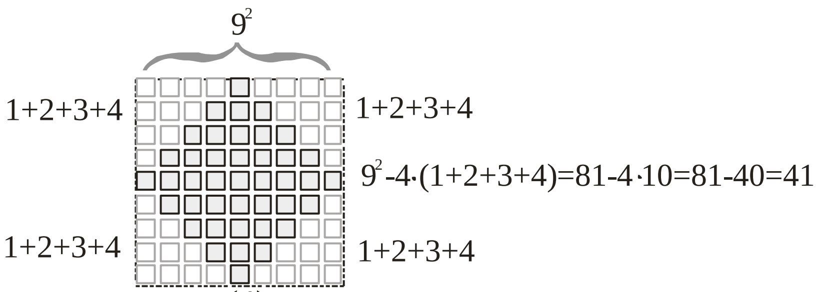
The parenthesis is equal to 100^2 as we remember from the gnomons of Pythagoras! If we want to calculate the sum of a set of odd numbers starting from 1, it is enough to find the plurality and raise the number to the square. In the sum 1+3+5+...+197+199 there are 100 numbers. So, if we add them we will find 100^2. The general rule is:

$$2\cdot[1+3+5+\ldots+(2n-1)]+(2n+1)=2n^2+2n+1 \text{ squares.}$$

The calculation of the parenthesis would be impossible without recourse to prior knowledge: the sum of $1+3+5+...+2n-1$ is equal to n^2. This historical and cultural knowledge is associated with the gnomons of Pythagoras.

2. The removal from the square: $(2n+1)^2-4(1+2+\ldots+n)=(2n+1)^2-2n(n+1)$

According to this method we imagine a large square of squares that contains figure (4). Its side is equal to the middle row of figure (4) that contains the most of small squares. The composition of figure (4) is obtained by subtracting the four triangles of squares which occupy the four corners of the large square that we had imaged in the beginning. For the 4th figure we have:



(4)

Having solved the problem for shape (4) we went on to the solution for shape (100) and shape (n).

In figure (1) we imagined a large square with side 3. We removed 1 square from each corner. So, we found: 3^2-4×1=9-4=5. As a result, figure (1) will have 5 squares. Similarly, shape (2) will have: 5^2-4×(1+2)=25-12=13 squares. We removed from every corner 3 squares (1+2), a total of 12. Figure (3) has: 7^2-4×(1+2+3)=49-24=25 squares. Figure (4) will have: 9^2-4×(1+2+3+4)=81-40=41 squares. Shape (n) contains: $(2n+1)^2$-4×(1+2+...+n) squares. The sum 1+2+3+...+n is equal to n (n+1)/2. We know this from the handshakes' problem that we mentioned before. Eventually, for 100th shape we have:

$$(101)^2-4\cdot(1+2+\ldots+100)=(101)^2-2\cdot 100\cdot 101=40.401-20.200=20.201$$

For the general solution we imagine a large square with a side 2n+1 and we subtract from it four equal triangles of squares (four equal sums of the form 1+2+3+...+n). So, the solution led to the following formula:

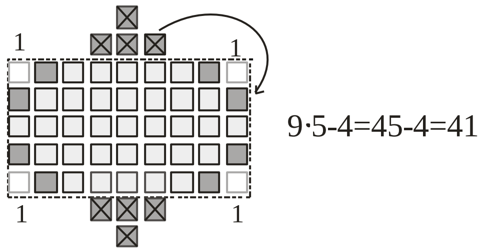
$$(2n+1)^2-4(1+2+3+\ldots+n)=(2n+1)^2-2n(n+1)$$

For the solution, the original geometric figure was modified, the handshake problem was applied and all the necessary arithmetic and algebraic operations were used.

3. Reconstructions of the figure: $(2n+1)(n+1)-n$ or $(n+1)^2+n^2$

Shape (4) is deconstructed and restructured in various ways. Below we present two methods of cutting and rebuilding the original shape.

The first method of reconstruction: *For figure (4) we constructed a rectangle with dimensions of 9×5. But in the corners, 4 squares are missing. We moved the four squares that are above and the four squares that are located below and after that we placed them on the left and on the right of the corners (1square will be missing from each corner). So, to complete a rectangle of dimensions 9×5 we still need to put 1 square on each corner. Figure (4) has 41 squares.*



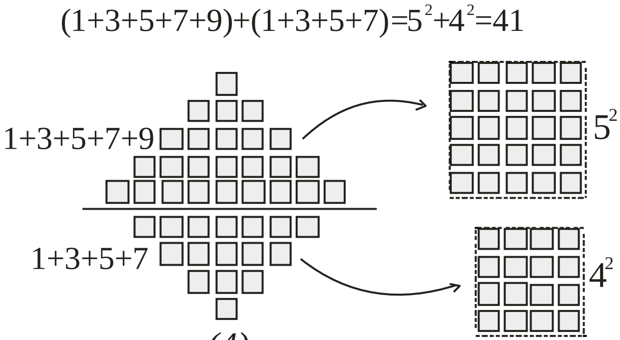
(4)

Figure (5) from the rectangle with dimensions 11×6, 5 squares are missing. Figure (5) contains 11×6-5=61 squares. Then, the 100th figure will have 201×101-100=20.301-100=20.201 squares. The nth figure will have (2n+1)(n+1)-ν squares. Consequently:

$$(2n+1)(n+1)-n=2n^2+2n+n+1-n=2n^2+2n+1=(n^2+2n+1)+n^2=(n+1)^2+n^2.$$

The second method of reconstruction: *We split figure (4) into triangles of squares. Each of these triangles can be transformed into a large square. We presented in the previous problem a single triangle of squares. If we have a second triangle under the 1^{st} one, then we will have two large squares of smaller squares. In figure (4) the total of 41 squares is divided into 25 squares in the upper triangle and 16 squares in the lower triangle. Our 1^{st} square has a side of 5 small squares and our 2^{nd} 4. They are:*

$$(1+3+5+7+9)+(1+3+5+7)=5^2+4^2=41 \text{ squares.}$$



(4)

In general, 100th figure will have 101^2+100^2 squares and the nth has to contain a total of $(n+1)^2+n^2$ squares.

All the previous methods in the problem of geometric pattern of quadratic type as well as our modifications, the geometric figure and their numerical relationships, helped us grasp a commonality and find an algebraic rule that allows an immediate determination of any term in the sequence.

Conclusions

The results of this study reveal that we solved each of these two problems of quadratic geometric patterns finding multiple solutions. The solutions we discovered are algebraic generalizations that began by considering on what we did with the geometric shape. We split the shapes and reorganized them so creating new shapes. The original geometric idea was the starting point. After that, the arithmetic and algebraic methods followed. In the first problem and in the second we found three different ways of algebraic generalization: construction, deconstruction and reconstruction.

In both problems we were able to find out the relationships that are not immediately obvious, by distinguishing what is the same and what is different. We grasped commonalities from specific examples. Close observation helped us discover the mathematical rule and prove it. Undoubtedly, the pre-existing knowledge to the subject (such as triangle and square numbers, handshake problem, arithmetic sequences, gnomons of Pythagoras and problems with other patterns) helped us a lot. All of these were incorporated into our new challenges and facilitated the algebraic generalization.

The algebraic generalization of a finite sample of specific cases, the grasping of the pattern and the formulation of a rule were fundamental mathematical activities that play a vital role in finding algebraic relationships. The use of numerical examples gave us the necessary cases that were needed to discover the hidden rule. By observing closely these specific examples we derived ideas for an infinite number of invisible and undefined cases, finding the general solution to the problem.

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