AN OPEN PROBLEM:
TRACING THE ANGLE BISECTOR

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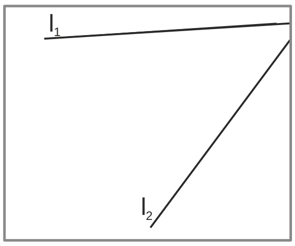
ABSTRACT

In this paper we will deal with the multiple solutions to an open problem of the Euclidean geometry. Its formulation is the following: "Two straight lines l_1 and l_2 intersect outside of a piece of paper, at a point O. Find as many methods as you can of tracing the angle bisector, formed by l_1 and l_2 without going outside of the sheet". The previous problem is solved with multiple ways that refer to a wide range of theorems, geometric concepts and demonstrative methods. The originality of conjectures, the wealth of ideas, the variety of geometrical figures and the numerous combinations of constructions reveal the beauty and the investigative fertility of the problem. The different ways are classified in four categories. As soon as we have solved the problem using a variety of different methods, we have created something original. The joy of discovery is a unique experience and we would like to share it with you.

Introduction

Over the years, problem solving has been recognised as one of the hallmarks of mathematics. According to George Polya (1887-1985), Hungarian, one of the greatest mathematicians of the 20th century, "solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable". "Open" problems are problems solved with multiple strategies that lead to different correct results or have several interpretations. We have chosen an interesting open problem of the Euclidean Geometry and not just a routine problem. In the Greek Secondary schools the tradition of the Euclidean Geometry is diachronic; it is a particularity that can be called ethnic mathematics. The problem is following:

THE PROBLEM: *Two straight lines l_1 and l_2 intersect outside of a piece of paper, in a point O. Find as many methods as you can of tracing the angle bisector, formed by l_1 and l_2 without going outside of the sheet.*

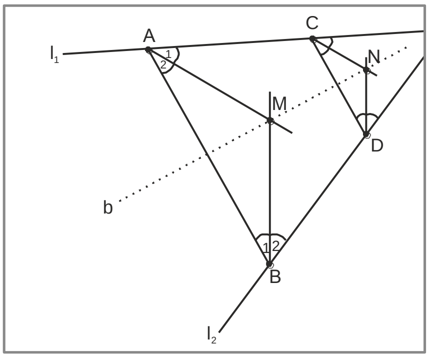


We present certain solutions of the problem that we discovered. The ways that we found are classified in the following categories.

First category: finding two points lying on the angle bisector

1st way

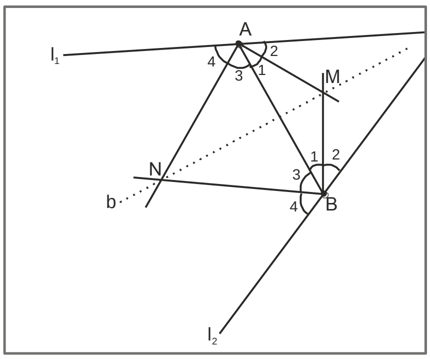
Let A and B be two arbitrary points on l_1 and l_2 respectively. Then draw the line segment AB. Let O be the point of intersection of l_1 and l_2, out of piece of paper. In the hypothetical triangle AOB we construct the interior angle bisectors of $\hat{A}$ and $\hat{B}$, which intersect at M. This means that the third angle bisector of the triangle AOB must also pass through M.



Similarly, we connect two other arbitrary points C and D, on l_1 and l_2 respectively, and we obtain the segment CD. In the hypothetical triangle COD we construct the interior angle bisectors of $\hat{C}$ and $\hat{D}$ just like we previously did, that intersect at the incenter of the triangle COD. In the same way we conclude that N lies on the angle bisector of $A\hat{O}B$ (just like M does). This means that MN is the angle bisector of $A\hat{O}B$.

Remarks: Instead of the determination of the incenters M and N of triangles AOB and COD respectively, we could find the excenter of the same triangles by tracing the exterior angle bisectors of $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. Similarly the exterior angle bicector of $\hat{A}$ meets the exterior angle bicector of $\hat{B}$ at M and the exterior angle bicector of $\hat{C}$ meets the exterior angle bicector of $\hat{D}$ at N. As a result MN is the angle bicector of $A\hat{O}B$. It is also possible to determine the incenter of the triangle AOB and the excenter of COD or vice versa. Finally, M and N can be determined by constructing a line perpendicular to the exterior angle bisectors of $\hat{A}$ and $\hat{B}$ (respectively $\hat{C}$ and $\hat{D}$).

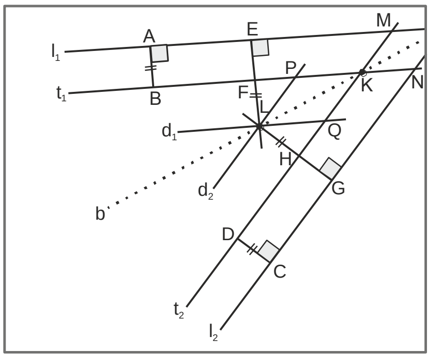
2nd way



Let A be an arbitrary point on l_1 and B a point on l_2. We join them and imagine a point O out of the piece of paper where l_1 and l_2 intersect forming a hypothetical triangle ABO. We then construct the interior angle bisectors of $\hat{A}$ and $\hat{B}$ which intersect at M. Similarly, we trace the exterior angle bisectors of $\hat{A}$ and $\hat{B}$ of the triangle AOB that intersect at N (the excenter of AOB). MN will then determine the angle bisector of $l_1\hat{O}l_2$. Alternatively, N could be defined as the point of intersection of the perpendicular on AM and the perpendicular on MB, the exterior angle bisector of $\hat{A}$ is always to the perpendicular interior angle bisectors of $\hat{A}$ (theorem).

3rd way

Let O be the point where l_1 and l_2 meet. We construct two straight lines t_1 and t_2 that intersect at a point K in the interior of $l_1\hat{O}l_2$ such that : t_1 // l_1, t_2 // l_2 and AB=CD. Their point of intersection K is a point that lies on the angle bisector of $l_1\hat{O}l_2$ because it is equidistant from l_1 and l_2. Similarly, we construct d_1// l_1 and d_2// l_2. Then their point of intersection L is a point of the angle bisector of $l_1\hat{O}l_2$. The points K and L define the angle bisector of $l_1\hat{O}l_2$.

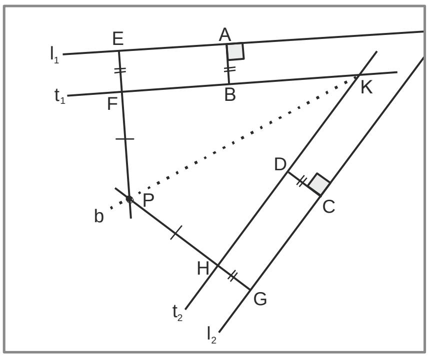


Remark: Quadrilaterals KPLQ and KMON are rhombus (parallelograms with equidistant opposite sides). Diagonals KL and KO are on the same line and form the required angle bisector of $l_1\hat{O}l_2$.

Second category: localisation of a point that lies on the required angle bisector and of a straight line congruent or parallel to it

1st way

We construct two straight lines t_1 and t_2 , in the interior of the angle formed by l_1 and l_2 such that: t_1 // l_1, t_2 // l_2, AB=CD and l_1 and l_2 intersect at K. We then trace the angle bisector Kb of $t_1\hat{K}t_2$. We make the assumption that Kb is also the angle

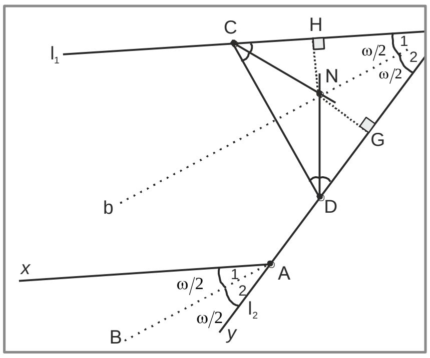


bisector of $l_1\hat{O}l_2$.

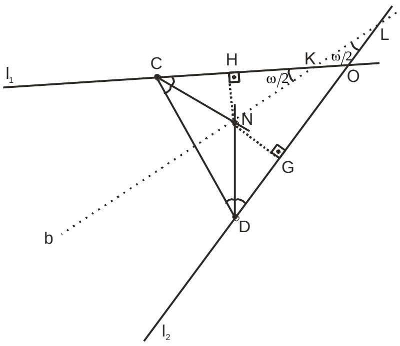
That is valid because any point P of Kb is equidistant not only from t_1 and t_2, but also from the sides of the angle $l_1\hat{O}l_2$. (PF+FE=PH+HG and so PE=PG).

Consequently, if we obtain the angle bisector of $t_1\hat{K}t_2$ then this coincides with the angle bisector of $l_1\hat{O}l_2$.

2nd way



We construct a straight line Ax// l_1, where A lies on l_2. Suppose, that O is the point of intersection of l_1 and l_2. The formed angle $x\hat{A}y$ is equal to $l_1\hat{O}l_2 = \hat{\omega}$ as corresponding angles. We trace the angle bisector of $x\hat{A}y$. The considered angle bisector *b* of $l_1\hat{O}l_2$ is parallel to AB because $\hat{O}_2 = \hat{A}_2 = \frac{\hat{\omega}}{2}$ (corresponding angles).

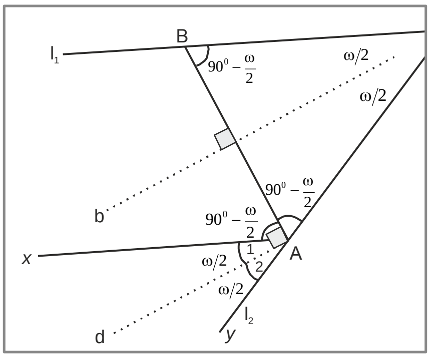


With one of the ways that were mentioned before we determine a point of the required angle bisector (for example N incenter of triangle OCD). From the point N we trace N*b*//AB.

So Nb is the angle bisector of $l_1\hat{O}l_2$ and passes from O. This occurs because assuming that Nb cuts l_1 at K and l_2 at L then from the equal right-angled triangles NHK and NGL we conclude NK=NL=NO.

Third category: finding a line segment forming an isosceles triangle) and construction of the perpendicular bisector

1st way



From an arbitrary point A lying on l_2 we construct $Ax//l_1$. Then the angle $x\hat{A}y$ is equal to the angle $l_1\hat{O}l_2=\hat{\omega}$ as corresponding angles, where O is the hypothetical point of intersection of l_1 and l_2.

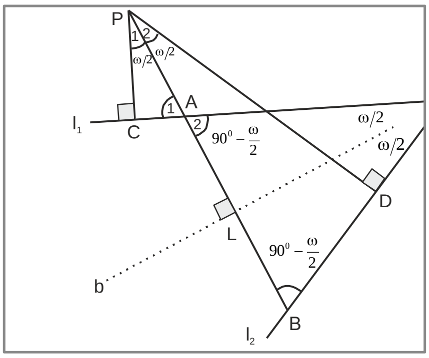
We then trace the angle bisector A*d* of $x\hat{A}y$

Then $\hat{A}_1=\hat{A}_2=\frac{\hat{\omega}}{2}$.

We construct a straight line $AB\perp Ad$ (where B lies on l_1) , which is the angle bisector of the adjacent angle of $x\hat{A}y$. So $B\hat{A}O=B\hat{A}x=90^0-\frac{\hat{\omega}}{2}$. From triangle ABO we have that: $A\hat{B}O=180^0-(90^0-\frac{\hat{\omega}}{2})-\hat{\omega}=90^0-\frac{\hat{\omega}}{2}$, thus the triangle ABO is isosceles with BO=OA. Consequently, the perpendicular bisector of AB is also the angle bisector of $l_1\hat{O}l_2$

2nd way

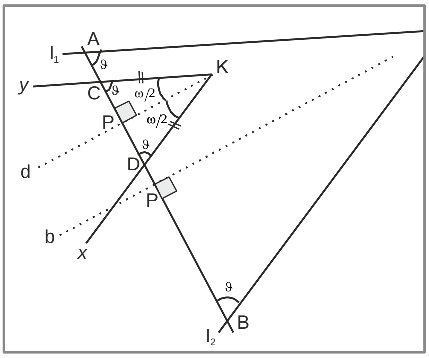
Let l_1 O $l_2=\omega$. From an arbitrary point P we construct $PC\perp l_1$ and $PD\perp l_2$. Hence $C\hat{P}D=\hat{\omega}$ (they have perpendicular sides). We trace the angle bisector PA of $C\hat{P}D$, that intersects l_2 at B. We have: $\hat{A}_1=\hat{A}_2=90^0-\frac{\hat{\omega}}{2}$. The triangle AOB is isosceles as $A\hat{B}O=180^0-(90^0-\frac{\hat{\omega}}{2})-\hat{\omega}=90^0-\frac{\hat{\omega}}{2}$.



Consequently the perpendicular bisector of AB coincides with the angle bisector of l_1 O l_2 , as AOB is isosceles.

Remark: As $PB \perp Ob$, if we have defined a point on Ob with one of the previous methods (for example incenter N) it is enough to trace from this point the perpendicular to PB. This is the angle bisector of $\hat{O}$.

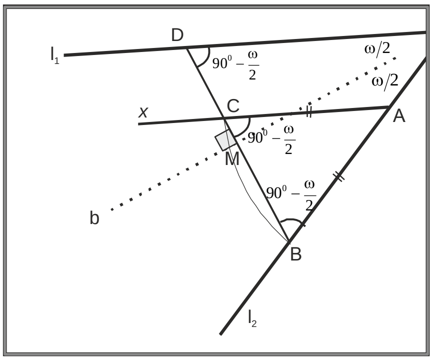
3rd way



From an arbitrary point K in the interior of $l_1\hat{O}l_2$ (or exterior) we construct parallels lines to the sides of the angle (Ky// l_1 and Kx//l_2). We have xKy= l_1 O l_2 =ω (parallel sides). Afterwards, by tracing a circle K an of arbitrary radius R we construct the isosceles triangle KCD. So yKx =l_1 O l_2 =ω, so yKd=$\frac{\hat{\omega}}{2}$ and hence: $\hat{\theta} = 90^0 - \frac{\hat{\omega}}{2}$. The line CD cuts l_1 at A and l_2 at B and the formed triangle AOB turns out to be isosceles as $\hat{A} = \hat{C} = \hat{\theta}$ and $\hat{B} = \hat{D} = \hat{\theta}$, so $\hat{A} = \hat{B} = \hat{\theta}$. As a result, the perpendicular bicector of AB is the angle bisector of $l_1\hat{O}l_2$ as well.

Remark: Instead of the isosceles triangle KCD we trace AB⊥K*d*. Then $B\hat{A}O = 90^0 - \frac{\hat{\omega}}{2}$ and $A\hat{B}O = 90^0 - \frac{\hat{\omega}}{2}$, thus the triangle ABO is isosceles and all previous results are valid about tracing the angle bisector.

4th way



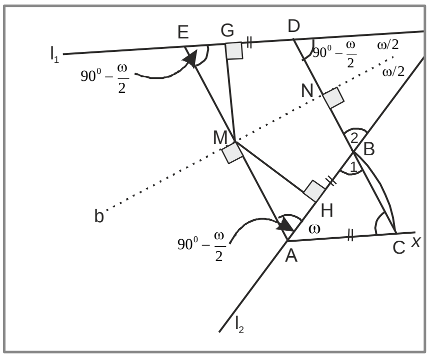
We construct $Ax \parallel l_1$ and then we trace a circle center A of an arbitrary radius R. So it follows that CA=BA=R and as a result $C\overset{\Delta}{A}B$ is isosceles. Therefore $A\hat{B}C + B\hat{C}A = x\hat{A}O \Leftrightarrow 2A\hat{B}C = x\hat{A}O = 180 - \hat{\omega}$ (as $x\hat{A}B = l_1\hat{O}l_2 = \hat{\omega}$). The line CB then cuts l_1 at D, So: $A\hat{C}B = A\hat{B}C$ and $A\hat{C}B = O\hat{D}B$, So: $O\hat{D}B = O\hat{B}D \Leftrightarrow O\overset{\Delta}{B}D$ is isosceles and we trace the perpendicular bisector of BD which coincides with the angle bisector of $l_1\hat{O}l_2$.

Fourth category: using properties of isosceles trapezium, rhombus, parallelogram and circumscribed quadrilaterals

1st way

From an arbitrary point A on l_2 we trace Ax// l_1. We construct isosceles triangle ABC (AB=AC) in the exterior angle $l_1\hat{O}l_2 = \hat{\omega}$. The line BC cuts l_1 at D. We then trace AE//BD. Hence: AB=DE (ACDE parallelogram) and ABDE is an isosceles trapezium.

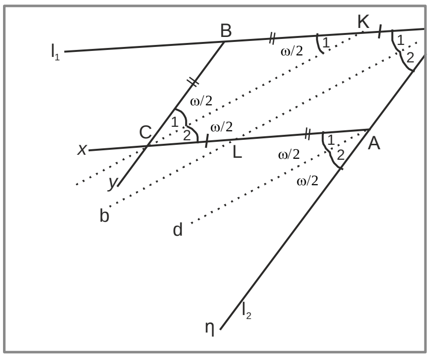
Therefore: $\hat{C} = \hat{B}_1 = \hat{B}_2 = \hat{D} = \hat{E} = \hat{A} = 90^0 - \frac{\hat{\omega}}{2}$ and $x\hat{A}B = \hat{\omega}$. The straight line passing through the midpoints M, N of EA and DB respectively is the required angle bisector of the angle $\hat{O}$. This occurs because as the right-angled triangles MEG and MHA are equal, M is equidistant from the sides of the angle $l_1\hat{O}l_2$. Similarly N is equidistant from the sides of the same angle so we have placed our angle bisector.



Remark: The isosceles triangle ABC is possible to be construct in the interior of $l_1\hat{O}l_2$.

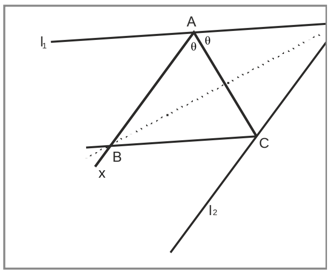
2nd way

From an arbitrary point A on l_2 we trace Ax// l_1. Then $x\hat{A}\eta = l_1\hat{O}l_2 = \hat{\omega}$ as corresponding angles, where O is the hypothetical point of intersection of l_1 and l_2. We trace the angle bisector of $x\hat{A}\eta$. Then $\hat{A}_1 = \hat{A}_2 = \frac{\hat{\omega}}{2}$. From an arbitrary point B on l_1 we construct By// l_2. BOAC is a parallelogram and therefore $\hat{C} = \hat{O} = \hat{\omega}$. We trace the angle bisector of the angle $\hat{C}$. The triangle BCK, where CB=BK is isosceles and so $\hat{C} = \hat{K}_1 = \frac{\hat{\omega}}{2}$. We consider AL=BK,



so CL=//KO and hence CLOK is a parallelogram. Then $\hat{C}_2 = \hat{O}_1 = \hat{O}_2 = \frac{\hat{\omega}}{2}$, namely LO is the angle bisector of $l_1\hat{O}l_2$.

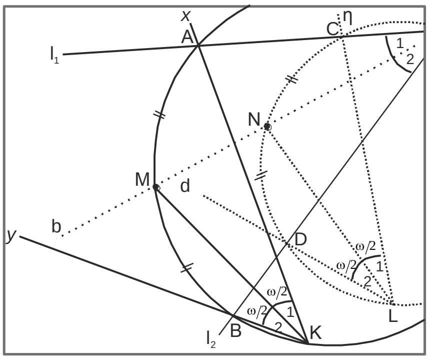
3rd way



From an arbitrary point A on l_1 we trace Ax// l_2. We also construct the angle bisector of $O\hat{A}x$, that intersects l_2 at C. Then we trace from C a line parallel to l_1 that cuts Ax at B. ABCO is a parallelogram because the opposite sides are parallels. Also: a diagonal (AC) bisects an angle ($X\hat{A}O$), so ABCO is a rhombus. As a result, to construct the angle bisector of the required angle we can trace the perpendicular bisector of AC, as OA=OC.

Remark: The parallelogram in which the opposite sides are equidistant is a rhombus (it can be proved with equal triangles).

4th way

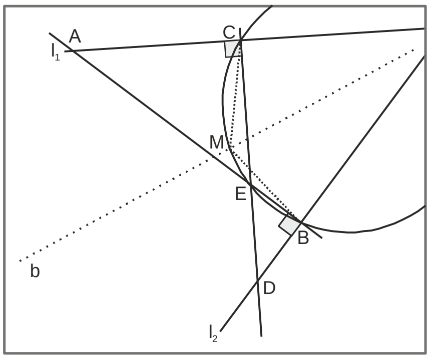


We trace an arbitrary angle $x\hat{K}y = \hat{\omega}$, Kx intersects l_1 at A and *Ky* intersects l_2 at B. We construct the circumscribed circle about the triangle ABK. The quadrilateral OKBA is circumscribable because O and K are angles at the circumference standing on the same arc AB. We construct the angle bisector of $x\hat{K}y$, which intersects the circle at M. As $\hat{K}_1 = \hat{K}_2 = \frac{\hat{\omega}}{2}$, $AM = MB$

Consequently, M is a point of the angle bisector of $l_1\hat{O}l_2$. In the same way we determine a second point N. Then MN is the required angle bisector.

5th way

We construct CD⊥ l_1 and AB⊥ l_2 such that A,C lie on l_1 and B,D lie on l_2, CD and AB intersect at E. We construct the circumscribed circle about the triangle BEC. The quadrilateral OCEB is circumscribable because $\hat{C} + \hat{B} = 180^0$. Provided that M is the midpoint of the arc CB then M is a point on the angle bisector of $l_1\hat{O}l_2$ this is the case as equal angles at the circumference of a circle stand on the same arc. In the same way we determine a second point N. Then MN is the required angle bisector.



Remark: It is not necessary that the angles C and B are right angles. It is sufficient to be supplementary angles that have theirs vertices on the sides of $l_1\hat{O}l_2$.

Conclusions

Solving this problem engaged us in a task for which the solution method was not known in advance for this reason we had to approach the problem through a scientific method: try, examine, conjecture, experiment, prove. The previous problem is solved with multiple ways that refer to a range of theorems, geometric concepts and demonstrative methods. We believe that there is no "best" strategy to solve it and that all the mentioned strategies are worthwhile. The originality of numerous conjectures, the wealth of ideas, the variety of geometrical figures and the combinations of constructions reveal the beauty and the investigative fertility of the problem. Although the focus of all these required a significant amount of effort, we are convinced that the problem is inexhaustible and we look forward to new interesting solutions. We are open to new ideas and approaches.

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