Statement of the problem

implicitization of curves, surfaces, hypersurfaces

1. Algebraic Geometry

2. Practical Applications

   Geometric Modeling
   Graphics
   Computer Aided Geometric Design
   CAD

parameterization (inverse problem)

The choice between the **implicit** and the **parametric** representations of a geometric object depends heavily on the particular nature of the application.

A parameterization of a geometric object in a space of dimension n can be given by a set of equations as follows:

$$x_1 = f_1(t_1, \ldots, t_k), \ldots, x_n = f_n(t_1, \ldots, t_k)$$

with $t_1, \ldots, t_k$ parameters,

and $f_1, \ldots, f_n$
polynomials, rational or trigonometric functions, functions involving square roots, etc.

$n = 2$ curves

$n = 3$ surfaces

$n \geq 4$ hypersurfaces

Implicitization

The implicitization problem consists in computing a polynomial cartesian (implicit) equation

$$p(x_1, \ldots, x_n) = 0,$$

of the geometric object described by the parametric equations, which satisfies

$$p(f_1(t_1, \ldots, t_k), \ldots, f_n(t_1, \ldots, t_k)) = 0,$$

for all values of the parameters $t_1, \ldots, t_k$.

Parameterization

find parametric equations, given the implicit equations

these two problems are not always solvable

e.g. logarithmic spiral

$$x = \alpha \cos\theta \exp^{\beta\theta}, y = \alpha \sin\theta \exp^{\beta\theta}$$

Description of the algorithm

Input:	Parametric equations for specific n, k.
Output:	Cartesian (implicit) equations of degree m.
Step 1:	construct the row vector v of all power products of total degree up to m in the vars $x_1, \ldots, x_n$.
Step 2:	compute the matrix $M = v^t \cdot v$.
Step 3:	substitute $x_1, \ldots, x_n$ by their parametric representation, in the matrix M.
Step 4:	integrate the elements of the matrix M successively over each parameter $t_1, \ldots, t_k$.
Step 5:	compute a basis of the nullspace of G
Step 6:	if the basis is empty then there is no implicit equation of degree m else implicit equations are given as $v \cdot nv$

Examples I (Curves)

Consider parametric equations of the unit circle:

$$x = \frac{1}{\sqrt{t^2+1}},\ y = \frac{t}{\sqrt{t^2+1}}$$

Construct the vector $v = [1, x, y, x^2, x\ \ y, y^2]$ and form the 6×6 matrix $M = v^t \cdot v$.

Substitute the parametric equations into the matrix and integrate for $t \in [0, 1]$.

Obtain a singular matrix of rank 5 with nullspace generated by $[-1, 0, 0, 1, 0, 1]$.

$$-1 + x^2 + y^2 = 0.$$

Examples II (Space Curves)

Consider parametric equations of the trefoil knot:

$$x = \left(\sqrt{2} + \cos(2\,t)\right)\cos(3\,t),$$

$$y = \left(\sqrt{2} + \cos(2\,t)\right)\sin(3\,t),\ \ z = \sin(2\,t)$$

In degree 4, we obtain a sparse 35×35 matrix with a nullspace of dimension 4. (integrations done in $[0, \pi]$)

$$x^4 + x^2y^2 + \tfrac{3}{2}\,x^2z^2 + 2\,\sqrt{2}x^2z^2 + \tfrac{1}{2}\,y^2z^2 + 2\,\sqrt{2}y^2z^2 + \tfrac{1}{2}\,z^4 + 2\,\sqrt{2}xyz - \tfrac{9}{2}\,x^2 - \sqrt{2}x^2 - \tfrac{3}{2}\,y^2 - \sqrt{2}y^2 + z^2 + \tfrac{1}{2}$$

$$x^4 - y^4 + z^2x^2 + 4\,z^2\sqrt{2}x^2 - 3\,x^2 - 2\,\sqrt{2}x^2 + 4\,\sqrt{2}zxy + 4\,z^2\sqrt{2}y^2 - 2\,\sqrt{2}y^2 + 3\,y^2 - z^2y^2$$

Examples III (Hypersurfaces)

Consider parametric equations of a hypersurface:

$$x = t + s + r, y = ts + sr + rt,$$

$$z = tsr, w = t^4 + s^4 + r^4$$

In degree $\mathbf{4}$, we obtain a sparse $\mathbf{70 \times 70}$ matrix with a nullspace of dimension $\mathbf{1}$. (integrations done in $[\mathbf{0}, \mathbf{1}]$)

$$-w + x^4 + 2\,y^2 - 4\,yx^2 + 4\,zx.$$

Examples IV (Families of ...)

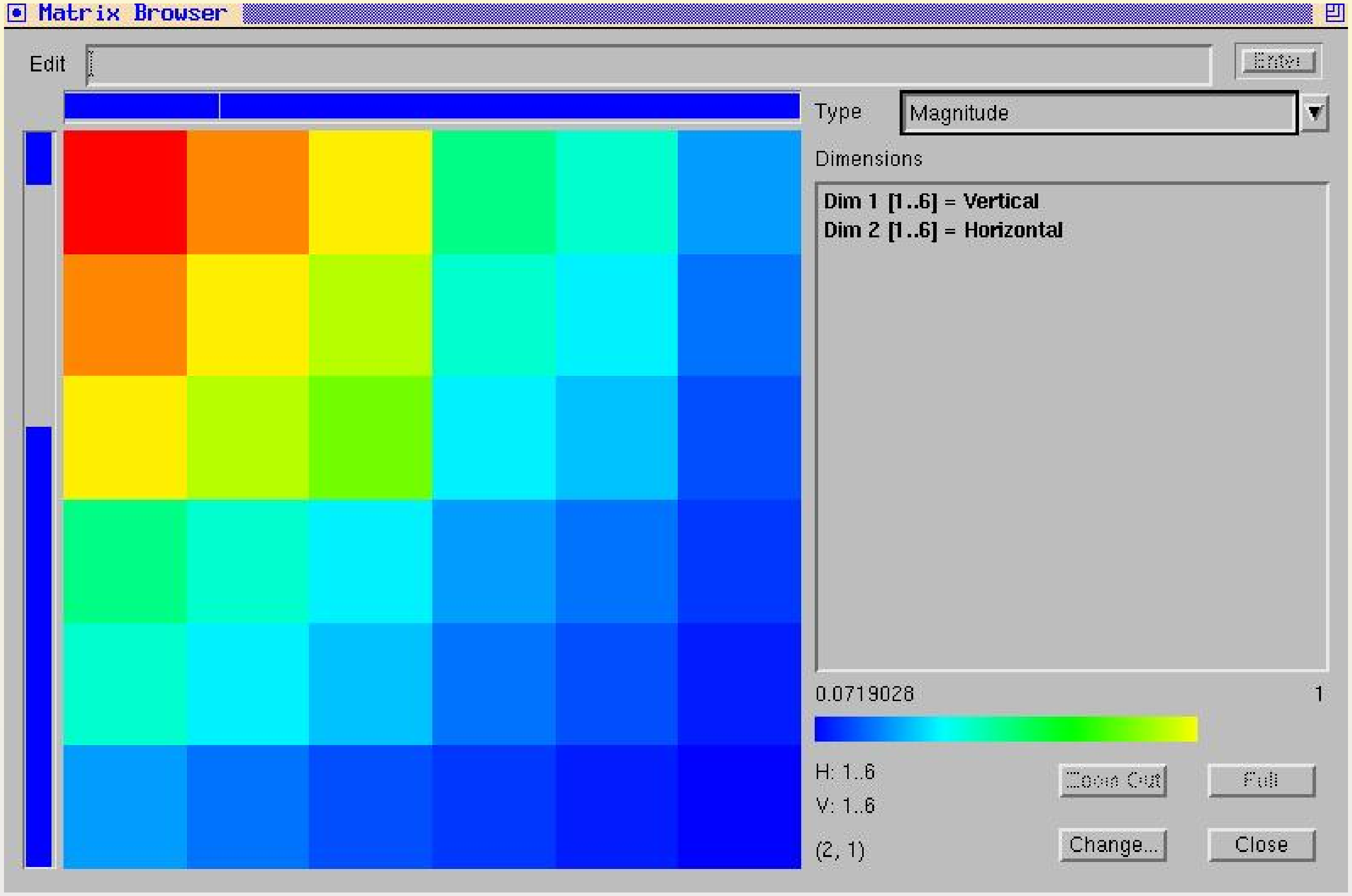
Consider a family of curves indexed by a parameter a defined by the following rational parametric equations:

$$x = \frac{t\left(a-t^2\right)}{\left(1+t^2\right)^2}, \qquad y = \frac{t^2\left(a-t^2\right)}{\left(1+t^2\right)^2}$$

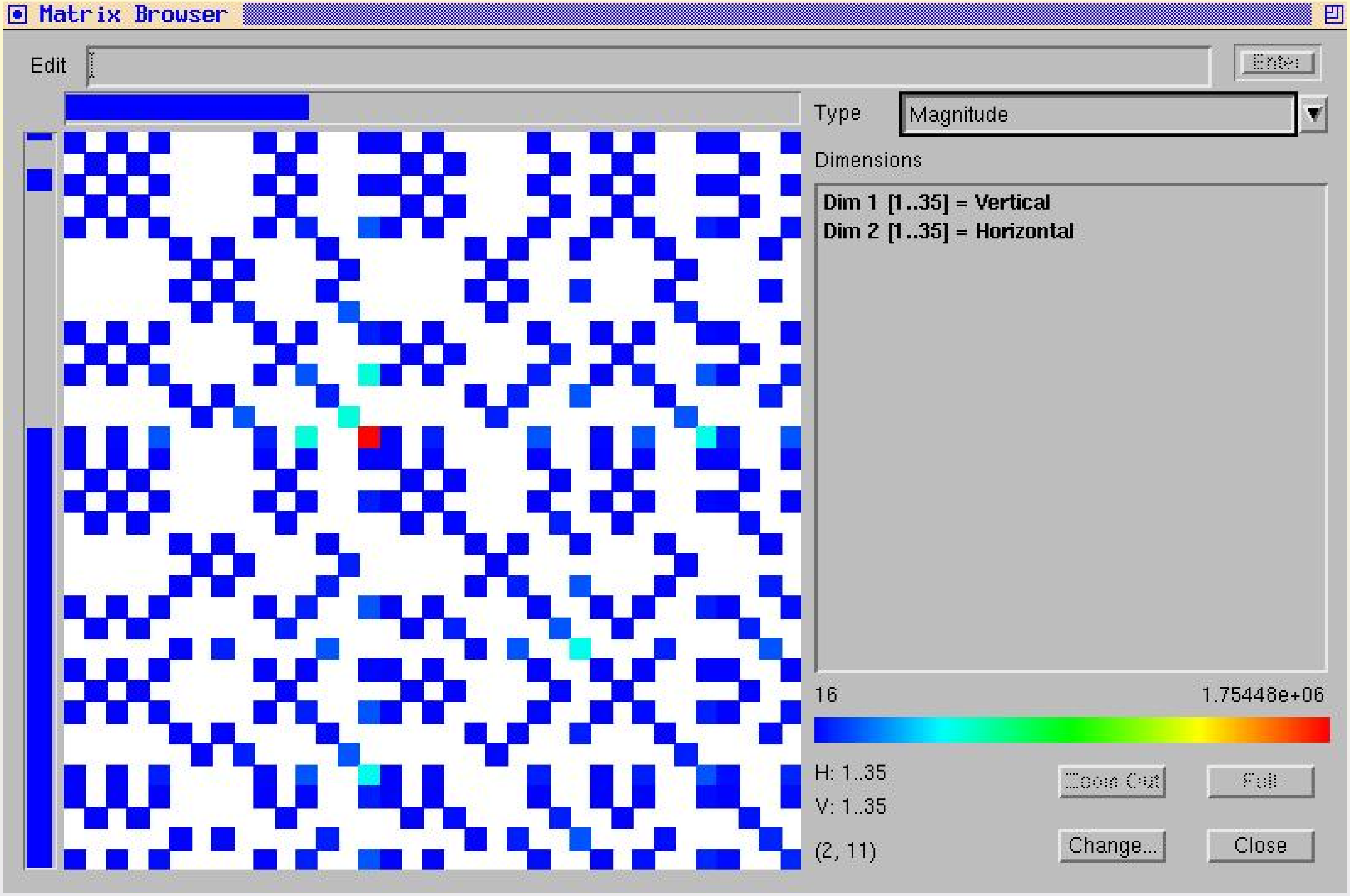
Compute the cartesian equation for some values of a and by extrapolation we have that the general monoparametric cartesian equation for this family of curves is:

$$x^4 - a\,yx^2 + 2\,x^2y^2 + y^3 + y^4 = 0.$$

Structure of the Implicitization Matrices



Unit Circle: $x = \dfrac{1}{\sqrt{t^2+1}}, y = \dfrac{t}{\sqrt{t^2+1}}$



Buchberger example (1961) $x^4 - yz = 0$

Descartes Folium $x^3 + y^3 = 3xy$ (different orderings)