Spaces and questions

Misha GROMOV

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§ I. Dawn of Space.

Our Euclidean intuition, probably, inherited from ancient primates, might have grown out of the first seeds of space in the motor control systems of early animals who were brought up to sea and then to land by the Cambrian explosion half a billion years ago. Primates' brain had been lingering for 30-40 million years. Suddenly, in a flash of one million years, it exploded into growth under relentless pressure of the sexual-social competition and sprouted a massive neocortex (70 % neurons in humans) with an inexplicable capability for language, sequential reasoning and generation of mathematical ideas. Then Man came and laid down the space on papyrus in a string of axioms, lemmas and theorems around 300 B.C. in Alexandria.

Projected to words, brain's space began to evolve by dropping, modifying and generalizing its axioms. First fell the Parallel Postulate: Gauss, Schweikart, Lobachevski*, Bolyai (who else?) came to the conclusion that *there is a unique non-trivial one-parameter deformation of the metric on* $\mathbb{R}^3$ *keeping the space fully homogeneous.***

It is believed, Gauss, who convinced himself in the validity of hyperbolic geometry somewhere between 1808 and 1818, was disconcerted by the absence of a Euclidean realization of the hyperbolic plane H^2. By that time, he must have had a clear picture of geometry of surfaces in $\mathbb{R}^3$ (exposed in his "Disquisitones circa superficies curvas" in 1827) where the (intrinsic) distance between two points on a surface is defined as the length of the shortest (better to say "infimal") curve *in* the surface between these points. (This idea must have been imprinted by Nature in the brain as most animals routinely choose shortest cuts on rugged terrains.) Gauss discovered the following powerful efficient criterion for isometry between surfaces, distinguishing, for example, a piece of a round sphere $S^2 \subset \mathbb{R}^3$ from an arbitrarily bent sheet of paper (retaining its intrinsic Euclideaness under bendings).

Map a surface $S \subset \mathbb{R}^3$ to the unit sphere S^2 by taking the vectors $\overline{\nu}(s) \in S^2$ parallel to the unit normal vectors $\nu(s),\ s \in S$.
If S is C^2-smooth, the Gauss map $G : S \to S^2$ for $s \mapsto \overline{\nu}(s)$ is C^1 and its Jacobian, i.e. the infinitesimal area distortion, comes with a non-ambiguous sign (since the directions of ν's give coherent orientations to S and S^2) and so S appears with a real function, called *Gauss curvature* $K(s) \underset{def}{=} \mathrm{Jac}\, G(s)$.

Theorema Egregium. *Every isometry between surfaces, say $f : S \to S_1$, preserves Gauss curvature, $K(f(s_1)) = K(s_1)$ for all $s_1 \in S_1$.*

For example, the plane has $K \equiv 0$ (as the Gauss map is constant) and so it is not (even locally) isometric to the unit sphere where $K \equiv 1$ (for the Gauss map

*Accidentally, the first mathematics' teacher of Gauss ($\approx$ 1790), Johann Martin Bartels, later on became the teacher of Lobachevski ($\approx$ 1810) in Kazan.

**A metric space X is *fully homogeneous* if every partial isometry $X \supset \Delta \leftrightarrow \Delta' \subset X$ extends to a full isometry of X (as for Euclid's triangles with equal sides in $\mathbb{R}^2$).

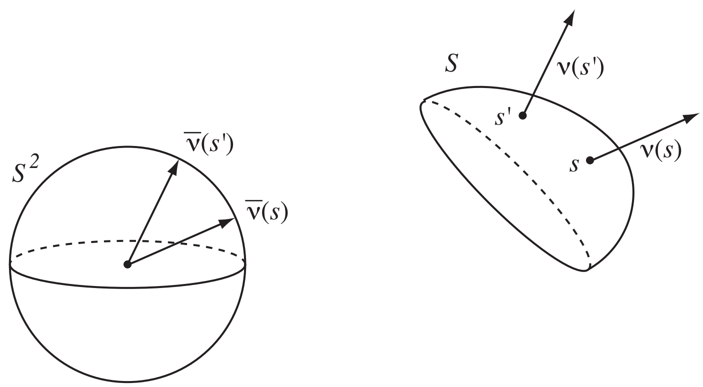

Figure 1

is identity on S^2). More generally, no strictly convex surface is locally isometric to a saddle surface, such as the graph of the function $z = xy$ for instance, since strict convexity makes $K > 0$ while saddle points have $K \leq 0$.

Gauss was well aware of the fact that the hyperbolic plane H^2 would have constant negative curvature if it were realized by a surface in $\mathbb{R}^3$. But he could not find such a surface! In fact, there are (relatively) small pieces of surfaces with $K = -1$ in $\mathbb{R}^3$ investigated by Beltrami in 1868 and it is hard to believe Gauss missed them; but he definitely could not realize the whole H^2 by a C^2-surface in $\mathbb{R}^3$ (as is precluded by a theorem of Hilbert (1901). This could be why (besides his timidity in the face of Kantian's guard of Trilobite's intuition)* Gauss refrained from publishing his discovery.

Probably, Gauss would have been delighted to learn (maybe he knew it?) that the flat Lorenz-Minkowski "metric" $dx^2 + dy^2 - dz^2$ on $\mathbb{R}^{2,1} = \mathbb{R}^3$ induces a true positive metric on the sphere

$$S^2_- = \left\{x, y, z \mid x^2 + y^2 - z^2 = -1\right\} ,$$

where each of the two components of S^2_- (one is where $z > 0$ and the other with $z < 0$) is isometric to H^2 and where the orthogonal group $O(2,1)$, (i.e. the linear group preserving the quadratic form $x^2 + y^2 - z^2$) acts on these two H^2's by (hyperbolic) isometrics.

There is no comparable embedding of H^2 into any $\mathbb{R}^N$ (though H^2 admits a rather contorted isometric C^∞-immersion to $\mathbb{R}^5$ (to $\mathbb{R}^4$?) and, incredibly, an isometric C^1-embedding into $\mathbb{R}^3$) but it admits an embedding into the Hilbert space, say $f : H^2 \to \mathbb{R}^\infty$, where the induced intrinsic metric is the hyperbolic one, where all isometries of H^2 uniquely extend to those of $\mathbb{R}^\infty$ and such that

$$\mathrm{dist}_{\mathbb{R}^\infty}(f(x),\ f(y)) = \ \sqrt{\mathrm{dist}_{H^2}\, x, y)} + \delta(dist(x, y))$$

*Of all people, had he been alive, Kant himself could have been able to assimilate, if not accept, the non-Euclidean idea.

with *bounded* function $\delta(d)$ (where one can find f with $\delta(d) = 0$, but this will be *not* isometric in our sense as it blows the lengths of all curves in H^2 to infinity). Similar embeddings exist for metric trees as well as for real and complex hyperbolic spaces of all dimensions but not for other irreducible symmetric spaces of non-compact type. (This is easy for trees: arrange a given tree in $\mathbb{R}^\infty$, such that its edges become *all* mutually orthogonal and have prescribed lengths.)

Summary. Surfaces in $\mathbb{R}^3$ provide us with a large easily accessible *pool of metric spaces*: take a domain in $\mathbb{R}^2$, smoothly map it into $\mathbb{R}^3$ and, voilà, you have the induced Riemannian metric in your lap. Then study the isometry problem for surfaces by looking at *metric invariants* (curvature in the above discussion), relate them to *standard spaces* ($\mathbb{R}^N$, $\mathbb{R}^\infty$, $\mathbb{R}^{2,1}$), and consider *interesting* (to whom?) *classes* of surfaces, e.g. those with $K > 0$ and with $K < 0$.

Remark and references. (a) It seems to me that the reverence for human intuition and introspective soul searching stand in the way to any attempt to understand how the brain does mathematics. Hopefully, experience of natural scientists may lead us to a meaningful model (a provisional one at this stage, say in the spirit of Kanerva's idea of distributed memory, see [Kan]).

(b) Our allusions to the history of mathematics are borrowed from [Klein], [Newm] and [Vasi].

(c) Little is known of what kind of maps $S \to S^2$ can serve as Gauss maps G of *complete* surfaces in $\mathbb{R}^3$. For example, given a domain $U \subset S^2$, one may ask whether there exists an *oriented closed immersed* (i.e. with possible selfintersections) surface $S \subset \mathbb{R}^3$ with $G(S) \subset U$ (where $U \neq S^2$ forces S to be topologically the 2-torus). This now appears to me a typical misguided "natural" question; yet I have not lost hope it may have a revealing solution (compare 2.4.4. in [$\mathrm{Gro}_{\mathrm{PDR}}$]).

(d) It is unknown if every surface with a C^∞-smooth Riemannian metric can be isometrically C^∞-immersed into $\mathbb{R}^4$. (Another "natural" question?) But isometric immersions into high dimensional spaces are pretty well understood (see 2.4.9 – 2.4.11 and Part 3 in [$\mathrm{Gro}_{\mathrm{PDR}}$]).

(e) The above equivariant embedding $H^2 \to \mathbb{R}^\infty$ tells us that the isometry group $\mathrm{Iso}H^2 = PSL_2\mathbb{R}$ is *a-T-menable*, opposite to *Kazhdan's property T* (defined in (A) of § V) satisfied by the majority of groups. (*A-T*-menability generalizes amenability. This property was first recognized by Haagerup, I presume, who used different terminology.)

§ II. Spirit of Riemann.

The *triangle inequality* is not always easy to verify for a given function in two variables $d : X \times X \to \mathbb{R}_+$ as it is a non-local property of d on X; thus one cannot create metrics at will. Yet, an arbitrary metric d on a connected space can be made "local" by replacing it by the supremal metric d^+ agreeing with d on an "arbitrary fine" covering of X (as we pass, for example, from the restriction of the Euclidean metric on a submanifold, e.g. a surface, in $\mathbb{R}^N$, to the induced Riemannian, or *intrinsic*, metric). More generally, following Riemann (Habilitationsschrift, June 10, 1854) one starts with an arbitrary field g of Euclidean metrics on a domain $U \subset \mathbb{R}^n$, i.e. a continuous map $u \mapsto g_u$ from U to the $\frac{n(n+1)}{2}$-dimensional space G of positive definite quadratic forms on $\mathbb{R}^n$. One measures distances in small neighbourhoods $U_\varepsilon(u) \subset U,\ u \in U$, by setting $d_{u,\varepsilon}(u', u'') = \|u' - u''\|_u = (g_u(u' - u'',\, u' - u''))^{\frac{1}{2}}$ for $u',\, u' \in U_\varepsilon$ and then defines the Riemannian (geodesic) distance dist_g on U as the supremal metric for these $d_{u,\varepsilon}$ as $\varepsilon \to 0$. Finally, Riemannian manifolds V appear as metric spaces locally isometric to the above U's. (The latter step is sleek but hard to implement. Try, for example, to show that the induced (intrinsic) metric on a smooth submanifold $V \subset \mathbb{R}^N$ is Riemannian in our sense.)

The magic power of this definition is due to the infinitesimal kinship of "Riemannian" to "Euclidean". If V is smooth (i.e. all g's on U's are at least C^2) then locally near each point v, one can represent V by a neighbourhood U of the origin $0 \in \mathbb{R}^n$ with $v \mapsto 0$, so that the corresponding g on U agrees with the Euclidean (i.e. constant in v) metric $g_0 = g_0(x, y) = \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ up to the *first* order,

$$g_u = g_0 + \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 g(0)}{\partial u_i \partial u_j}\right) u_i u_j + \dots\,,$$

i.e. with the first order Taylor terms missing and where, moreover, only $\frac{n^2(n^2-1)}{12}$ terms among $\left(\frac{n(n+1)}{2}\right)^2$ second derivatives $\frac{\partial^2 g_{\mu\nu}(0)}{\partial u_i \partial u_j}$ do not vanish. The resulting $\frac{n^2(n^2-1)}{12}$ functions on U, when properly organized, make the *Riemann curvature tensor* of V (which reduces to the Gauss curvature for $n = 2$) measuring the deviation of (V, g) from flatness (i.e. Euclideaness).

The (polylinear) algebraic structure built into g allows a full fledged analysis on (V, g), such as the Laplace-Hodge operator, potential theory etc. This turned out to be useful for particular classes of manifolds distinguished by additional (global, local or infinitesimal) symmetry, where the major achievements coming to one's mind are:

- Hodge decomposition on the cohomology of Kähler manifolds V and a similar (non-linear) structure on the spaces of representations of $\pi_1(V)$.

- Existence of Einstein metrics on Kähler manifolds with algebra-geometric consequences.

- Spectral analysis on locally symmetric (Bruhat-Tits and adelic as well as Riemannian) spaces leading, for instance, to various cohomology vanishing theorems, T-property (with applications to expanders) and (after delinearization) to super-rigidity of lattices in semi-simple Lie groups.

The linear analysis on *general* Riemannian manifolds pivots around the Atiyah-Singer-Dirac operator and the index (Riemann-Roch) theorem(s). These originated from Gelfand's question (raised in the late fifties) aiming at an explicit topological formula for the index of an elliptic operator (which is easily seen to be deformation invariant) and became a central theme in mathematics starting from the 1963 paper by Atiyah and Singer.

The *non-linear* Riemannian analysis on general V's followed for the most part the classical tradition concentrating around elliptic variational problems with major advances in the existence and regularity of solutions: minimal subvarieties, harmonic maps, etc. The most visible "external" application, in my view, concerns manifolds with *positive scalar curvature* – the subject motivated by problems (and ideas) coming from general relativity – resolved by Schoen and Yau with a use of minimal hypersurfaces.

Manifolds of each dimension *two, three* and *four* make worlds of their own, richer in structure than all we know so far about $n \geq 5$.

In dimension two we possess the Cauchy-Riemann equations and are guided by the beacon of the Riemann mapping theorem, the crown jewel of differential geometry.

The four-dimensional pecularity starts with algebra: the orthogonal group $O(4)$ locally decomposes into two $O(3)$. This allows one to split (or rather to square-root) certain natural (for the $O(4)$-symmetry) non-linear second order operators in a way similar to how we extract the Cauchy-Riemann $\overline{\partial}$ from the boringly natural selfadjoint Laplacian Δ on $\mathbb{R}^2$. The resulting first order operators (may) have non-zero indices and satisfy a kind of non-linear index theorem discovered by Donaldson in 1983 for the Yang-Mills and then extended to the Seiberg-Witten equation. (Both equations were first written down by physists, according to the 20th century lore.)

Manifolds of dimension three borrow from their two- and four-dimensional neighbours: Thurston's construction of hyperbolic metrics on basic 3-manifolds relies on geometry of surfaces while Floer homology descends from Yang-Mills.

Shall we ever reach spaces beyond Riemann's imagination?

Remark and references. It will need hundreds of pages to account for the above forty lines. Here we limit ourselves to a few points.

(a) The Riemannian metric g naturally (i.e. functorially) defines *parallel transport* of vectors along smooth curves in V which is due to the absence of first derivatives in an appropriate Taylor expansion of g. This can be seen clearly for V realized in some $\mathbb{R}^N$ (which is not a hindrance according to the Cartan-Janet-Burstin-Nash isometric embedding theorem) where a family $X(t)$ of tangent vectors is parallel *in* V along our curve γ parametrized by $t \in \mathbb{R}$ iff the ordinary (Euclidean) derivative $\frac{dX(t)}{dt} \in \mathbb{R}^N$ is normal to V at $\gamma(t) \in V$ for all t. (This is independent of the isometric embedding $V \to \mathbb{R}^N$.) If a curve

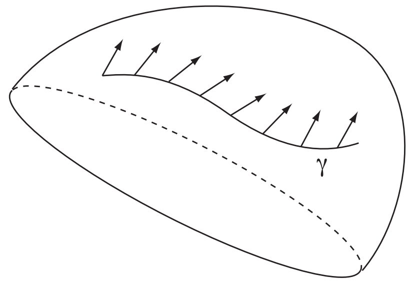

Figure 2

$\gamma : [0,1] \to V$ comes back making a loop (i.e. $\gamma(0) = \gamma(1)$), every tangent vector $X = X(0) \in T_v(V)$, $v = \gamma(0) \in V$, transforms to $\gamma_*(X) \overset{def}{=} X(1) \in T_v(V)$ and we obtain a homomorphism from the "group" of loops at v to the linear group of isometric automorphisms of the tangent space $T_v(V) = \mathbb{R}^N$, i.e. to the orthogonal group $O(n)$; its image $H \subset O(n)$ is called the *holonomy group* of V (which is independent of v for connected V). Generically, $H = O(n)$ ($SO(n)$ for orientable V), but sometimes H has positive codimension in $O(n)$. For example, $\dim H = 0$ iff V is locally Euclidean (Parallel Postulate is equivalent to $H = \{\mathrm{id}\}$) and if $V = V_1 \times V_2$, then $H = H_1 \times H_2 \subset O(n_1) \times O(n_2) \underset{\neq}{\subset} O(n)$ for $n_1, n_2 \neq 0$. Then there are several *discrete* series of *symmetric spaces* – monumental landmarks towering in the vastness of all Riemannian metrics $\mathbb{R}^n$, S^n, H^n, $\mathbb{C}P^n$, $SL(n)/SO(n)$... It is natural to think that these are essentially all V's with small holonomy, since $\operatorname{codim} H > 0$ implies a rather overdetermined system of P.D.E. for g (for example, $\dim H = 0 \Leftrightarrow$ curvature $(g) = 0$, i.e. $\frac{n^2(n^2-1)}{12}$ equations against mere $\frac{n(n+1)}{2}$ components of g. Yet flat metric exists!). But lo and behold: lots of even dimensional manifolds carry *Kähler metrics* where $H \subset U(n) \underset{\neq}{\subset} SO(2n)$. Just take a *complex analytic* submanifold V in $\mathbb{C}^N$ (or in $\mathbb{C}P^N$) and observe (which is obvious once being said) that the parallel transport in the induced metric preserves the complex structure in the tangent spaces. (This may be not so striking, perhaps, for those who have absorbed the impact of holomorphic functions, overdetermined by Cauchy-Riemann in many variables, but there are less expected beautiful exotic holonomy beasts predicted by Berger's classification and brought up to life by Briant, see [Bria].)

The Kähler world is tightly knit (unlike the full Riemannian universe) with deep functorial links between geometry and topology. For example, the first cohomology of a compact Kähler V comes by the way of a holomorphic (!) map to some complex torus, $V \to \mathbb{C}^d/\text{lattice}$ for $d = \frac{1}{2}\operatorname{rank} H^1(V)$. This extends to (non-Abelian) representations $\pi_1(V) \to GL(n)$ for $n \geq 2$ (Siu, Corlette, Simpson ..., see [A-B-C-K-T]) furnishing something like an "*unramified* non-Abelian Kählerian class field theory" (in the spirit of Langlands' program) but we have no (not even conjectural) picture of the "transcendental part" of $\pi_1(V)$

(killed by the profinite completion of π_1). Can, for example, π_1 (Kähler) have an unsolvable word problem? Is there an internal structure in the category of Kähler fundamental groups functorially reflecting the geometry of the Kähler category? (All *known* compact Kähler manifolds can be deformed to complex projective ones and it may be preferable to stay within the complex algebraic category, with no fear of ramifications, singularities and non-projectiveness.)

(b) There exists a unique (up to normalization) second order differential operator $\mathbf{S}$ from the space of positive definite quadratic differential forms (Riemann metrics) g on V to the space of functions $V \to \mathbb{R}$ with the following two properties.

$\mathbf{S}$ is Diff-equivariant for the natural action of diffeomorphisms of V on both spaces.

$\mathbf{S}$ is linear in the second derivatives of g (being a linear combination of components of the full curvature tensor).

Then $\mathbf{S}(g)$ (or $\mathbf{S}(V)$) is named the *scalar curvature* of (V, g) with the customary normalization $\mathbf{S}\ (S^n) = n(n-1)$.

If $n = 2$, $\mathbf{S}$ coincides with Gauss curvature, it is additive for products, $\mathbf{S}(V_1 \times V_2) = \mathbf{S}(V_1)+\mathbf{S}(V_2)$ and it scales as g^{-1}, i.e. $\mathbf{S}(\lambda g) = \lambda^{-1}\ \mathbf{S}(g)$ for $\lambda > 0$.

The following question proved to be more to the point than one could expect.

What is the geometric and topological structure of manifolds with $\mathbf{S} > 0$*?* (This comes from general relativity as $\mathbf{S} > 0$ on world sheets reflects positivity of energy.)

The condition $\mathbf{S} > 0$ appears quite plastic for $n \geq 3$, where one can rather freely manipulate metrics g keeping $\mathbf{S}(g) > 0$, e.g. performing geometric surgery; besides, every compact V_0 turns into V with $\mathbf{S}(V) > 0$ when multiplied by a small round sphere of dimensions ≥ 2. Yet this plasticity has its limits: Lichnerowicz found in 1963 a rather subtle topological obstruction ($\widehat{A}(V) = 0$ if V is spin) with the use of the index theorem. Then Schoen and Yau approached the problem in 1979 from another angle (linked to ideas in general relativity) and proved, among other things, that n-tori (at least for $n \leq 7$) carry no metrics with $\mathbf{S} > 0$ thus answering a question by Geroch. Inspired by this, we revived with Blaine Lawson in 1980 Lichnerowicz' idea, combined it with the Lusztig-Mistchenko approach to the *Novikov conjecture* on homotopy invariance of Pontryagin classes of non-simply connected manifolds and found out that the bulk of the topological obstructions for $\mathbf{S} > 0$ comes from a "limit on geometric size" of V induced by the inequality $\mathbf{S} > 0$ (similar to but more delicate than that for $K > 0$, see III).

Yet, the above question remains open with an extra mystery to settle: what do minimal hypersurfaces and the Dirac operator have in common? (Seemingly, nothing at all but they lead to almost identical structure results for $\mathbf{S} > 0$, see [Gro$_{\mathrm{PCMD}}$] for an introduction to these issues.)

It appears that an essential part of the difficulty in understanding $\mathbf{S} > 0$ (and the Novikov conjecture) is linked to the following simple minded question: *what is the minimal* $\lambda > 0$*, such that the unit sphere* $S^N_\infty(1)$ *in the Banach space* $\ell^N_\infty = (\mathbb{R}^N, \|x\| = \sup_i |x_i|)$ *admits a* λ*-Lipschitz map into the ordinary* n*-sphere* $S^N(1)$ *in* $\mathbb{R}^N = \ell^N_2$ *with non-zero degree?* Probably, $\lambda \to \infty$ for $N \to \infty$ (even if we stabilize to maps $S^N_\infty(1) \times S^M(R) \to S^{N\times M}(1)$ with arbitrarily large M and R) and this might indicate new ways of measuring "size of V" in the context of $\mathbf{S} > 0$ and the Novikov conjecture.

"Soft and hard". Geometric (and some non-geometric) spaces and categories (of maps, tensors, metrics, (sub)varieties...) can be ranked, albeit ambiguously, according to plasticity or flexibility of (the totality of) their members.

(1) Topology could appear flabby and structureless to Poincaré's contemporaries but when factored by homotopies (the very source of flexibility) it crystallizes to a rigid algebraic category as hard and symmetric as a diamond.

(2) Riemannian manifolds, as a whole, are shapeless and flexible, yet they abide "conservation laws" imposed by the Gauss-Bonnet-Chern identities. Deeper rigidity appears in the presence of elliptic operators extracting finite dimensional structure out of infinite dimensional depth of functional spaces. Also we start seeing structural rigidity (e.g. Cheeger compactness) by filtering metrics through the glasses of (say, sectional) curvature.

(3) Kähler metrics and algebraic varieties seem straight and rigid in the Riemannian landscape (never mind a dense set of Riemannian spaces appearing as real loci of complex algebraic ones) but they look softish in the eye of an algebraic geometer. He/she reinforces rigidity with the Calabi-Yau-Aubin theorem turning Kähler to Einstein-Kähler. (Nothing of the kind seems plausible in the full Riemannian category for $n > 4$.)

(4) Homogeneous, especially symmetric, spaces stand on the top of the geometric rigidity hierarchy (tempting one to q-deform them) and (sometimes hidden) symmetries govern integrable (regarded rigid) systems. (Softness in dynamics is associated with hyperbolicity.)

(5) Lattices Γ in semi-simple Lie groups grow in rigidity with dimension, passing the critical point at $\Gamma \subset SL_2(\mathbb{C})$, where they flourish in Thurston's hyperbolic land. A geometer unhappy with Mostow (over)rigidity for $n > 3$ is tempted to switch from lattices to (less condensed) subgroups with infinite covolumes and more balanced presentations (to dismay of a number theorist thriving on the full arithmetic symmetry of Γ). Most flexible among all groups are (generalized) small cancellation ones followed by higher dimensional hyperbolic groups while lattices and finite simple groups are top rigid. Vaguely similarly, the rigidity of Lie algebras increases with decrease of their growth culminating in Kac-Moody and finite dimensional algebras.

(6) Holomorphic functions on Stein manifolds V are relatively soft (Cartan's theory) as well as holomorphic maps $f : V \to W$ for homogeneous and elliptic

(i.e. with a kind of exponential spray) W by the (generalized) Grauert theorem allowing a homotopy of every continuous map $f_0 : V \to W$ to a holomorphic one. Holomorphic maps moderated by bounds on growth become more rigid (e.g. functions of finite order have essentially unique Weierstrass product decomposition). Algebraic maps, ordinarily rigid, sometimes turn soft, e.g. for high degree maps of curves to $\mathbb{P}^1$ by the Segal theorem. And Voevodski theory (if I interpret correctly what little I understood from his lecture) softens the category of algebraic varieties by injecting some kind of homotopies into these.

(7) **Big three**. There are three outstanding instances where striking structural patterns emerge from large and flexible geometric spaces: symplectic/contact, dimension 4 and $\mathbf{S} > 0$, conducted in all three cases by "Riemannian" and "elliptic". We met $\mathbf{S} > 0$ earlier (which seems least conceptually understood among the big three), symplectic and contact belong to Eliashberg and Hofer at this meeting (with "soft" versus "hard" discussed (in [Gro$_{\mathrm{SH}}$]) and nobody, alas, gave us a panorama of $n = 4$).

(8) **h-Principle**. Geometers believed from 1813 till 1954, since Cauchy (almost) proved rigidity of closed convex polyhedral surfaces in $\mathbb{R}^3$, that isometric immersions are essentially rigid. Then Nash defied everybody's intuition by showing that every smooth immersion of a Riemannian manifold $f_0 : V \to \mathbb{R}^N$ can be deformed, for $N - 2 \geq n = \dim V$, to a C^1-smooth (not C^2!) isometric $f : V \to \mathbb{R}^N$ with little limitation for this deformation, allowing one in particular, to freely C^1-deform all $V \subset \mathbb{R}^N$ keeping the induced (intrinsic) metric intact. (This is sheer madness from a hard-minded analyst's point of view as the N components of f satisfy $\frac{n(n+1)}{2}$ partial differential equations comprising an overdetermined system for $N < \frac{n(n+1)}{2}$, where one expects no solutions at all!) The following year (1955) Kuiper adjusted Nash's construction to $N = n + 1$ thus disproving C^1-rigidity of convex surfaces in $\mathbb{R}^3$.

Next, in 1958, Smale stunned the world by turning the sphere $S^2 \subset \mathbb{R}^3$ inside out. He did it not by exhibiting a particular (regular) homotopy (this was done later and only chosen few are able to follow it through) but by developing a homotopy theoretic approach used by Whitney for immersions of curves into $\mathbb{R}^2$. Then Hirsch incorporated Smale into the obstruction theory and showed that a continuous map $f_0 : V \to W$ can be homotoped into an immersion *if* the obvious necessary condition is satisfied: f_0 lifts to a fiberwise injective homomorphism of tangent bundles, $T(V) \to T(W)$, with the exception of the case of closed equidimensional manifolds V and W where the problem is by far more subtle.

It turned out that many spaces X of solutions of partial differential equations and inequalities abide the homotopy principle similar to that of Nash, Smale-Hirsch and Grauert: every such X is canonically homotopy equivalent to a space of continuous sections of some (jet) bundle naturally associated to X. (For example, the space of immersions $V \to W$ is homotopy equivalent to the space of fiberwise injective morphisms $T(V) \to T(W)$ by the Hirsch theorem.)

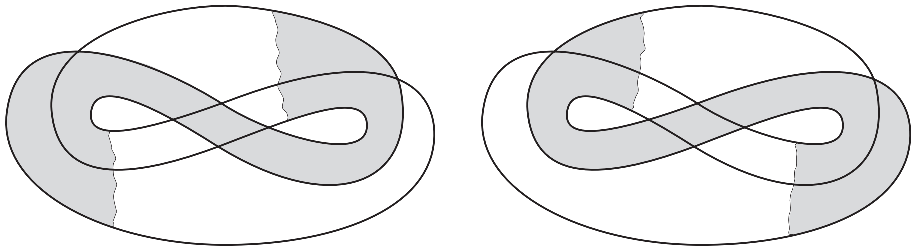

Figure 3

The geometry underlying the proof of the h-principle is shamefully simple in most cases: one creates little (essentially 1-dimensional, à la Whitney) wrinkles in maps $x \in X$ which are spread all over by homotopy and render X soft and flexible. But the outcome is often surprising as seen in the (intuitively inconceivable) Milnor's two *different* immersed disks in the plane with common boundary which come up with logical inevitability in Eliashberg's folding theorem.

Despite the growing array of spaces subjugated by the h-principle (see $[\mathrm{Gro}]_{\mathrm{PDR}}$, [Spr]) we do not know how far this principle (and softness in general) extends (e.g. for Gauss maps with a preassigned range, see I). Are there sources of softness not issuing from dimension one? Some encouraging signs come from Thurston's work on foliations, Gao-Lohkamp h-principle for metrics with Ricci < 0, and especially from Donaldson's construction of symplectic hypersurfaces (where "softness" is derived from a kind of "ampleness" not dissimilar in spirit to Segal's theorem, see (6)).

A tantalizing wish is to find new instances, besides the big three, where softness reaches its limits with something great happening at the boundary. Is there yet undiscovered life at the edge of chaos? Are we for ever bound to elliptic equations? If so, what are they? (There are few globally elliptic non-linear equations and no general classification. But even those we know, coming from Harvey-Lawson calibrations, remain mainly unexplored.) And if this wish does not come true we still can make living in soft spaces exploring their geometry (their topology is completely accounted for by the h-principle) as we do in anisotropic spaces (see III).

(9) Our "soft and hard" are not meant to reveal something profound about the nature of mathematics, but rather to predispose us to acceptance of geometric phenomena of various kinds. Besides, it is often more fruitful to regard "numbers", "symmetric spaces", "$\mathrm{Gal}\overline{\mathbb{Q}}/\mathbb{Q}$", "$SL_n(\mathbb{A})/SL_n(\mathbb{Q})$"... as "true mathematical entities" rather than descendants of our general "spaces", "groups", "algebras" etc. But one cannot help wondering how these perfect entities could originate and survive in the softly structured brain hastily assembled by blind evolution. Some basic point (scientific, not philosophical) seems to completely elude us.

Nature and naturality of questions. Here are (brief, incomplete, personal and ambiguous) remarks intended to make clearer, at least terminologically, the issues raised during discussions we had at the meeting.

"Natural" may refer to the structure or *nature of mathematics* (granted this exists for the sake of argument), or to "natural" for *human nature.* We divide the former into (pure) **m**athematics, **l**ogic and **p**hilosophy, and the latter, according to (internal or external) reward stimuli, into **i**ntellectual, **e**motional, and **s**ocial. **E**(motional) plays upper hand in human decision (and opinion) (except for a single man you might have a privilege to talk mathematics to) and in some people (Fermat, Riemann, Weil, Grothendieck) **i-e** naturally converges to **m-l-p**. But for most of us it is not easy to probe the future by conjecturally extrapolating mathematical structures beyond present point in time. How can we trust our mind overwhelmed by **i-e-s** ideas to come up with true **m-l-p** questions?(An **e-s**-minded sociologist would suggest looking at trends in fund distribution, comparable weights of authorities of schools and individuals and could be able to predict the influencial role of Hilbert's problems and Bourbaki, for example, without bothering to read a single line in there.) And "**i-e-s**-natural" does not make "a stupid question": the 4-color problem, by its sheer difficulty (and expectation for a structurally rewarding proof) has focused attention on graphs while the solution has clarified the perspective on the role of computers in mathematics. But this being unpredictable, and unrepeatable, cannot help us in **m-l-p**-evaluation of current problems which may look **i-e** deceptively 4-colored. (As for myself, I love unnatural, crazily unnatural problems but you stumble upon them so rarely!)

§ III. $K \gtrless 0$ and other metric stories.

What are "most Euclidean" Riemannian manifolds?

We have been already acquainted with the fully homogeneous spaces also called, for a good reason, (complete simply connected) *of constant curvature* K: the round S^n with $K = +1$, the *flat* $\mathbb{R}^n$ with $K = 0$, and the hyperbolic H^n with $K = -1$.* (Observe that λS^n and λH^n converge to $\mathbb{R}^n$ for $\lambda \to \infty$ in a natural sense, where $\lambda(X, \mathrm{dist}) \stackrel{def}{=} X(\lambda, \mathrm{dist})$ and $K(\lambda X) = \lambda^{-\frac{1}{2}} K(X)$, as is clearly seen, for example, for λ-scaled surfaces $X \subset \mathbb{R}^3$.) Now, somewhat perversely, we bring in topology and ask for *compact* manifolds with constant curvature, i.e. *locally* isometric to one of the above S^n, $\mathbb{R}^n$, or H^n. Letting S^n go, we start with the flat (i.e. $K = 0$) case and confirm that compact locally Euclidean manifolds exist: just take a lattice Λ in $\mathbb{R}^n$ (e.g. $\Lambda = \mathbb{Z}^n$) and look at the *torus* $T = \mathbb{R}^n / \Lambda^n$. Essentially, there is little else to see:

F-**theorem.** *Every compact flat manifold X is covered by a torus with the number of sheets bounded by a universal constant $k(n)$.*

*The fourth and the last fully homogeneous Riemannian space is $P^n = S^n / \{\pm 1\}$.

This sounds dry but it hides a little arithmetic germ on the bottom: *there is no regular* k*-gon with vertices in a lattice* Λ (e.g. $\Lambda = \mathbb{Z}^2$) *in* $\mathbb{R}^2$ *for* $k \geq 7$ (or for $k = 5$). Indeed, transported edges of such k-gone R would make a *smaller* regular $R' \subset \Lambda$ and the contradiction follows by iteration R'', R''', ...

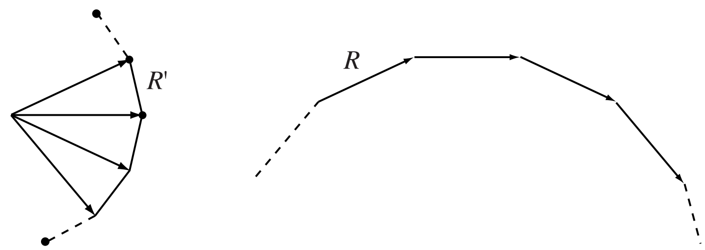

Figure 4

The tori T themselves stop looking flat as they all together make the marvellous moduli space $SO(n)\backslash SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ (of isometry classes of T's with $\mathrm{Vol}\, T = 1$) locally isometric to $SL_n(\mathbb{R})/SO(n)$ apart from mild (orbifold) singularities due to elements of finite order (< 7) in $SL_n(\mathbb{Z})$.

Turning to $K = -1$, we may start wondering if such spaces exist in a compact form at all. Then, for $n = 2$, we observe that the angles of *small* regular k-gones $R \subset H^2$ are almost the same as in $\mathbb{R}^2$ while *large* $R \subset H^2$ have almost zero angles: thus, by continuity, for every $k \geq 5$, there exists $R_\Box \subset H^2$ with 90° angles. We reflect H^2 in (the lines extending the) sides of $R_\Box$ and take the subgroup $\Gamma \subset \mathrm{Isom}(H^2)$ generated by these k reflections. This Γ is *discrete* on H^2 with $R_\Box$ serving as a fundamental domain similarly to the case of the square $R_\Box \subset \mathbb{R}^2$ and the quotient space H^2/Γ (equal R) becomes an honest manifold (rather than orbifold) if we take instead of Γ a subgroup $\Gamma' \subset \Gamma$ without torsion (which is not hard to find).

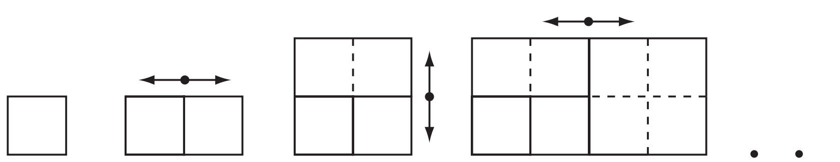
Figure 5

The same idea works for dodecahedra in H^3 and some other convex polyhedra in H^n for small n, but there are no compact hyperbolic reflection groups for large n (by a difficult theorem of Vinberg). The only (known) source of high dimensional Γ comes from arithmetics, essentially by intersecting $SO(n,1)$ somehow embedded into $SL_N(\mathbb{R})$ with $SL_N(\mathbb{Z})$ (where the orthogonal groups $SO(n,1)$ double covers the isometry group $PSO(n,1)$ of $H^n \subset \mathbb{R}^{n,1}$). Non-arithmetic Γ are especially plentiful for $n = 3$ by Thurston's theory and often have unexpected features, e.g. some $V = H^3/\Gamma$ fiber over S^1 (which is hard to imagine ever happening for large n). Moreover, the topological 3-manifolds fibered over S^2 *generically*, (i.e. for pseudo-Anosov monodromy) admit metrics with $K = -1$. Unbelievable – but is true by Thurston (who himself does not exclude that finite covers of most atorical V fiber over S^1; yet this remains open even for V with $K = -1$).

Alexandrov's spaces. *What are the most general (classes of) spaces similar to those with $K = \pm 1$?*

Alexandrov suggested an answer in 1955 by introducing spaces with $K \geq 0$, where the geodesic triangles have the sum of angles $\geq 2\pi$, and those with $K \leq 0$, where (at least small) triangles have it $\leq 2\pi$. But we take another, more functorial route departing from the following

Euclidean K-theorem. *Every 1-Lipschitz (i.e. distance non-increasing) map f_0 from a subset $\Delta \subset \mathbb{R}^n$ to some $\mathbb{R}^m$ admits a 1-Lipschitz extension $f : \mathbb{R}^n \to \mathbb{R}^m$ for all m, $n \leq \infty$.*

This is shown by constructing f point by point and looking at the worst case at each stage where extendability follows from an obvious generalization of the pretty little lemma:

Lemma. *Let Δ and Δ' in $S^{n-1} \subset \mathbb{R}^n$ be the sets of vertices of two simplices (inscribed into the sphere S^{n-1}) where the edges of Δ are correspondingly $\leq$ than those of Δ'. Then Δ is congruent to Δ' (i.e. $\leq \Rightarrow =$), provided Δ is not contained in a hemisphere.*

Now we say that a metric space X has $K \geq 0$, if every partial 1-Lipschitz map $X \supset \Delta \to \mathbb{R}^m$ extends to a 1-Lipschitz $f : X \to \mathbb{R}^m$, for all m, while $K \leq 0$ is defined with such extensions for $\mathbb{R}^n \supset \Delta \to X$, where in the case $K \leq 0$ one requires, besides the existence of an extension $\mathbb{R}^n \to X$, the *uniqueness* of this on the *convex hull* $\operatorname{Conv} \Delta \subset \mathbb{R}^n$ for $n = 1$, provided the starting map $\Delta \to X$ was isometric.

To make this worthwhile, one adds the metric completeness of X and the *locality property*: $\operatorname{dist}(x, x')$ should equal the infimal length of curves in X between x and x'. (Equivalently, there is a middle point $y \in X$ where $\operatorname{dist}(x, y) + \operatorname{dist}(y, x') = \operatorname{dist}(x, x')$.) Then, one arrives at the following elegant proposition justifying the definitions:

K-Theorem. *If $K(X) \geq 0$ and $K(Y) \leq 0$, then every partial 1-Lipschitz map $X \supset \Delta \to Y$ admits a 1-Lipschitz extension $X \to Y$.*

To apply this one needs examples of spaces with $K \gtrless 0$ and these are easier to observe with Alexandrov's definition. Fortunately, both definitions are equivalent and we have:

Complete (e.g. closed) convex hypersurfaces in $\mathbb{R}^n$ have $K \geq 0$, while saddle surfaces in $\mathbb{R}^2$ have (at least locally) $K \leq 0$. Symmetric spaces of compact type have $K \geq 0$ while those of non-compact type have $K \leq 0$.

Take a 2-dimensional polyhedron V assembled of convex Euclidean k-gones and observe that the link L of each vertex in V is a graph (i.e. 1-complex) with a length assigned to every edge e equal the k-gonal angle corresponding to e.

Then $K(V) \geq 0 \Leftrightarrow$ each L is isometric to a segment $\leq \pi$ or to a circle no longer than 2π.

$K(V) \leq 0 \Leftrightarrow$ all cycles in all L are longer than 2π and V is simply connected.

Finally, if X has locally $K \geq 0$, then so is true globally while globally $K \leq 0 \Leftrightarrow$ (locally $K \leq 0$) + ($\pi_1 = 0$).

The $\pi_1 = 0$ condition breaks the harmony (I guess it was upsetting Alexandrov) and brings confusion to the notion of $K \leq 0$ as the "local" and "global" meanings diverge. But in the end of the day the π_1-ripple makes the geometry of $K \leq 0$ much richer (and softer) than all we know of $K \geq 0$, since there are lots of spaces V with $K_{\mathrm{loc}} \leq 0$ (as is already seen in the 2-dimensional polyhedra) where the group theoretic study of $\pi_1(V)$ may rely on geometry (for example, in the Novikov conjecture). Besides, the global definition of $K \leq 0$ can be relaxed by, roughly, allowing λ-Lipschitz extension of partial 1-Lipschitz maps with $1 \leq \lambda \leq \mathrm{const} < \infty$ bringing along larger classes of (hyperbolic-like) spaces and groups where geometry and algebra are engaged in a meaningful conversation.

Anisotropic spaces. There is a class of metrics which can be analytically generated with the same case as the Riemannian ones; besides, we find among them spaces X in some way closer to $\mathbb{R}^n$ than S^n and H^n: these X are metrically homogeneous as well as *self-similar*, i.e. λX is isometric to X for all $\lambda > 0$. (In the Riemannian category there is nothing but $\mathbb{R}^n$ like that.)

A polarization on a smooth manifold V (e.g. $\mathbb{R}^n$ or a domain in $\mathbb{R}^n$) is a subbundle H of the tangent bundle $T(V)$, i.e. a field of m-planes on V, where for $1 \leq m \leq n-1$, $n = \dim V$ in the case at hand. Besides H we need an auxiliary Riemannian metric g on V but what matters is the restriction of g on H. We define $\mathrm{dist}(v, v') = \mathrm{dist}_{H,g}(v, v')$ by taking the infimum of g-lengths of piecewise smooth curves between v and v' which are chosen among H-*horizontal* curves, i.e. those which are *everywhere tangent* to H. It may happen that this distance is infinite (even for connected V) if some points admit no horizontal connecting paths between them, as it happens for *integrable* H, where $\mathrm{dist}(v, v') < \infty \Leftrightarrow v$, and v' lie in the same leaf of the foliation integrating H. This is not so bad as it seems but we want dist $< \infty$ at the moment and so we insist on the existence of a horizontal path between every two points in V. It is not hard to show that *generic* C^∞-smooth polarizations H do have this property for $m \geq 2$, where "generic" implies, in particular, that the space of "good" H's is open and dense in the space of all C^∞-polarizations on V.

A practical way for checking this is to take $m_+ \geq m$ vector fields tangent to H, and spanning it (these always exist), say $X_1, \ldots, X_{m_+}$. Then the sufficient criterion for our H-connectivity (also called *controllability*) is as follows: The successive commutors X_i, $[X_i, X_j]$, $[[X_i, X_j], X_k] \ldots$ of the fields span the tangent bundle $T(V)$.

The simplest instance of the above is the pair of the fields $X_i = \frac{\partial}{\partial x_1}$ and $X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$ in $\mathbb{R}^3$, where $[X_1, X_2] = -\frac{\partial}{\partial x_3}$. Here H can be presented as the kernel of the *standard contact form* $dx_3 + x_1 dx_2$ and, in fact, contact fields H are H-connected for all contact manifolds of dimension ≥ 3.

Next, look at a left invariant polarization H on a Lie group G defined by the left translates of a linear subspace h in the Lie algebra $L = L(G)$. It is not hard to see that the above holds $\Leftrightarrow$ there is no Lie subalgebra in L containing h besides L itself. Take, for instance, the 3-dimensional group *Heisenberg* G (homeomorphic to $\mathbb{R}^3$) with $L = L(G)$ generated by x_1, x_2, x_3, where $[x_1, x_2] = x_3$ and x_3 commutes with x_1 and x_2. Here one takes h spanned by x_2, x_3 and observe that this h is invariant under *automorphisms* of L defined by $x_1 \mapsto \lambda^2 x_1^2$, $x_2 \mapsto \lambda x_2$, $x_3 \mapsto \lambda x_3$ for all $\lambda > 0$. Then, for each left invariant metric g on G, the corresponding *automorphisms* $A_\lambda : G \to G$ preserves H and scale g *on* H by λ. Thus G is selfsimilar but quite different from $\mathbb{R}^3$. For example, the Hausdorff dimension of $(G, \mathrm{dist}_{H,g})$ is 4 rather than 3.

What are natural maps between the above non-isotropic spaces? Lipschitz maps do not serve here as well as in the Riemannian category since our new spaces are usually *not* mutually bi-Lipschitz equivalent. On the other hand typical spaces X and Y of the same topological dimension are locally Hölder homeomorphic with some positive exponent $\alpha < 1$ with a bound $\alpha > \alpha_n > 0$. But the optimal value of α for given spaces (or classes of those) remains unknown. A similar problem is that of finding sharp lower bound for the Hausdorff dimension of a subset $Y \subset X = (X, \mathrm{dist}_{H,g})$ by the topological dimension in terms of commutation properties of fields spanning H. (This is quite easy if Y is a *smooth* submanifold in X.)

The large scale geometry of $(X, \mathrm{dist}_{H,g})$ is rather close to the Riemannian geometry of (X, dist_g) and so H does not matter much. Conversely, the *local* geometry essentially depend on H (and very little on g) and the main open problems are basically local.

Concentrated *mm* **spaces.** Let X be a metric space which is also given a Borel measure μ, often assumed to be a probability measure, i.e. $\mu(X) = 1$. This μ may come out of the metric (e.g. the Riemannian measure, sometimes normalized to have $\mu(X) = 1$), but often μ has little to do with the original distance as, for example, the Gaussian measure on $\mathbb{R}^n$. We want to study (X, dist, μ) in a probabilistic fashion by thinking of functions f on X as random variables being concerned with their distributions, i.e. push-forward measures $f_*(\mu)$. Here is our

Basic problem. Given a map $f : X \to Y$, relate *metric* properties of f to the structure of the *measure* $f_*(\mu)$ on Y.

Here "metric" refers to how f distorts distance (expressed, for instance, by the Lipschitz constant $\dot{\lambda}(f)$), where we distinguish the case of functions, i.e. of $Y = \mathbb{R}$. On the probabilistic side we speak of "structure of $f^*(\mu)$" expressed entirely in terms of Y and where a typical question is how *concentrated* $f_*(\mu)$ is, i.e. how close it is to a point measure in Y.

Gaussian example. Take $\mathbb{R}^N$ for X with the measure $(\exp - \|x\|^2)dx$. *Then, every 1-Lipschitz function* $f : \mathbb{R}^N \to \mathbb{R}$ *is at least as much concentrated as the orthogonal projection* $f' : \mathbb{R}^N \to \mathbb{R}$. (There is a *1-Lipschitz* self-mapping $\mathbb{R} \to \mathbb{R}$ pushing forward $f'_*(\mu)$ to $f_*(\mu)$.)

A parallel example, where the geometry is seen clearer, is $X = S^N$ with the normalized Riemannian measure. Here again every 1-Lipschitz function $f : S^N \to \mathbb{R}$ is concentrated as much as the linear one; consequently, $f^*(\mu)$ converges, for $N \to \infty$, to a δ-measure on $\mathbb{R}$ (with the rate $\approx \sqrt{N}$).

The essence of the above concentration is the sharp contrast between the spread of the original measure on X (e.g. the distance between μ-random points in S^N is $\approx \pi/2$ for large N) and strong localization of $f_*(\mu)$ on Y (e.g. the characteristic distance on $\mathbb{R}$ with respect to $f_*(\mu)$ is $\approx 1/\sqrt{N}$ in the spherical case). The following definition is aimed to capture this phenomenon in the limit for $N \to \infty$ by interbreeding metric geometry with the ergodic theory (not quite as in the ergodic theorem where $f = f_N$ appears as the average of N transforms of a given f_0).

Let X be a (probability) measure space and $d : X \times X \to \mathbb{R}_+ \cup \infty$ a function satisfying the standard metric axioms except that we allow $d(x, x') = \infty$. In fact, we are keen at the (apparently absurd) situation of $d = \infty$ almost everywhere on X, moderated by the

Ergodicity axiom. For every $Y \subset X$ with $\mu(X) > 0$ the distance to Y,

$$d_Y(x) \underset{def}{=} \inf_{y \in Y} d(x, y)$$

is measurable and a.e. finite on X.

Example. Let X be a foliated measure space where each leaf is (measurably in $x \in X$) assigned a metric. Then we define

$$d(x, y) = \begin{cases} \infty \text{ if } x, y \text{ are not in the same leaf} \\ d(x, y) = \operatorname{dist}_L(x, y) \text{ if } x \text{ and } y \text{ lie in some leaf } L\,, \end{cases}$$

and observe that our ergodicity axiom amounts here to the ordinary ergodicity.

Next we distinguish *concentrated spaces* by insisting on the universal bound on the distances between subsets in X in terms of the measures of these subsets. Here

$$\operatorname{dist}(Y, Y') \overset{def}{=} \inf d(y, y') = \inf_{x \in Y'} d_Y(x)$$

and the bound is given by a function $C(a, a')$, $a, a' > 0$, (where C may go to infinity for $a, a' \to 0$), so that $\operatorname{dist}(Y, Y') \leq C(\mu(Y), \mu(Y'))$.

Examples. If X in the previous example is foliated (i.e. partitioned) into the orbits of an *amenable* group G acting on X, then the resulting d on X is, essentially, never concentrated. But if G has property T, then it is concentrated.

Let X_1, $X_2, \ldots$ be a sequence of Riemannian manifolds and X be the infinite Cartesian product, $X_1 \times X_2 \times \ldots$, where the "metric" between infinite sequences $x = (x_1, x_2 \ldots)$ and $y = (y_1, y_2, \ldots)$ is the Pythagorean one,

$$\mathrm{dist}(x,y) \stackrel{def}{=} \left(\mathrm{dist}^2(x_1,y_1) + \mathrm{dist}^2(x_2,y_2) + \ldots\right)^{\frac{1}{2}} .$$

What goes wrong here is that $\mathrm{dist}(x,y) = \infty$ for most x and y but we can tolerate this in the presence of the product measure μ on X coming from normalized Riemannian measures on X_i. If the first (sometimes called second) eigenvalues of (the Laplace operator Δ on) X_i are separated from zero, $\lambda_1(X_i) \geq \varepsilon > 0$, then the product is concentrated and, moreover, one can make a meaningful analysis (on functions and often on forms) on X. Furthermore, if $\lambda_1(X_i) \to \infty$ for $i \to \infty$, then Δ has discrete spectrum on X with finite multiplicity (computed by the usual formula for products). For instance, if X_i are n_i-spheres of radii R_i, then $\lambda_1(X_i) = n_i R_i^{-2}$ and their product is concentrated for $R_i \leq \varepsilon\sqrt{n_i}$ with extra benefits for $\sqrt{n_i}/R_i \to \infty$.

One can deform and modify products, retaining concentration, e.g. for projective limits of some towers of smooth fibration, such as the infinitely iterated unit tangent bundles of Riemannian manifolds.

Similarly to products, the spaces X of maps between Riemannian manifolds, $A \to B$, carry (many different) "foliated Hilbert manifold" structures which in the presence of measures (e.g. Wiener measure for 1-dimensional A) allow analysis on X.

Now comes a painful question: are these X good for anything? Do they possess a structural integrity or just encompass (many, but so what?) examples? A convincing theorem is to be proved yet.

Remarks and references.

(a) The F-theorem makes the core of Bieberbach's solution to a problem on Hilbert's list (N. 18, where the n-dimensional hyperbolic case is also mentioned and dismissed as adding little new to the results and methods of Fricke and Klein).

(b) Our definition of $K \gtrless 0$ is motivated by [La-Sch], where the authors prove the K-theorem for maps from spaces with $K \geq \lambda$ to those with $K \leq \lambda$ (in the sense of Alexandrov) under a mild restriction ruling out, for example, maps $S^n \to S^m$ for $m < n$, where K-property obviously fails to be true but, unfortunately, missing maps $S^n \to S^m$ for $m \geq n$ where it is known to be true (a conclusive version seems not hard but no published proof is available). Also, there is a Lipschitz extension result from arbitrary metric spaces to those with $K \leq 0$ (due to Lang, Pavlovic and Schroeder).

(c) The theory of spaces with $K \geq 0$ (as well as with $K \leq \lambda < \infty$) is by now well developed (see [Per]) and structurally attractive. Yet it suffers from the lack of a systematic process of generating such spaces apart from convex hypersurfaces (and despite several general constructions: products, quotients, spherical suspension ...). Also, there is no serious link with other branches of geometry, not even with the local theory of Banach spaces, and there are few theorems (K-theorem is a happy exception) where the conclusion is harder to verify (in the available examples) than the assumptions (as is unfortunately frequent in the global Riemannian geometry). A possible way of enriching (and softening) $K \geq 0$ is letting $n \to \infty$ (and taking $n = \infty$ seriously) as is done for Banach spaces. (See [Berg] for a broader perspective, [Pet] for a most recent account and [$\mathrm{Gro}_{\mathrm{SGM}}$] for a pedestrian guide to curvature.)

(d) **Ricci**. Analytically speaking, the most natural of curvatures is the Ricci tensor that is the quadratic differential form on V associated to g via a (essentially unique) Diff-invariant second order quasi-linear differential operator (like **S**) denoted $\mathrm{Ri} = \mathrm{Ri}(g)$. Manifolds with positive (definite) Ri generalize those with $K \geq 0$ but they admit no simple metric description as their essential features involve the Riemannian measure on V, e.g. R-balls in V with $\mathrm{Ri} \geq 0$ have smaller volumes than the Euclidean ones. (If $K \geq 0$ is motivated by convexity, then $\mathrm{Ri} \geq 0$ can be traced to positive mean curvature of hypersurfaces.) Thus it remains unclear how far the idea of $\mathrm{Ri} \geq 0$ extends beyond the Riemannian category: what are admissible singularities and what happens for $n = \infty$? (See [$\mathrm{Gro}_{\mathrm{SGM}}$] and [G-L-P] for an introduction and [Che-Co] for the present state of art.)

Encouraged by $\mathrm{Ri} \geq 0$ one turns to $\mathrm{Ri} \leq 0$ formally generalizing $K \leq 0$ but the naive logic does not work: *every metric can be approximated by those with* $\mathrm{Ri} < 0$ *by the Lohkamp h-principle* and no hard structural geometry exist.

(e) **Einstein and the forlorn quest for the best metric.** It is geometers' dream (first articulated by Heinz Hopf, I believe) to find a canonical metric g_{best} on a given smooth manifold V so that all topology of V will be captured by geometry. This happened to come true for surfaces as all of them carry (almost unique) metrics of constant curvature and is predicted for $n = 3$ by Thurston's geometrization conjecture. Also, there is a glimpse of hope for $n = 4$ (Einstein, self-dual) but no trace of g_{best} has ever been found for $n \geq 5$. What is the reason for this? Let us take some (energy) function E on the space $\mathcal{G}$ of metrics, say built of the full curvature tensor, something like $\int (\mathrm{Curv}(g))^{\frac{n}{2}} dv$ (where the exponent $n/2$ makes the integral scale invariant). Imagine, the gradient flow of E brings all of $\mathcal{G}$ to a "nice" subspace $\mathcal{G}_{\mathrm{best}} \subset \mathcal{G}$ (ideally, a single point or something not very large anyway). Then the group Diff V would act on $\mathcal{G}_{\mathrm{best}}$ (as all we do should be Diff-invariant) with compact isotropy subgroups (we assume V is compact at the moment), e.g. if $\mathcal{G}_{\mathrm{best}}$ consisted of a single point, then Diff V would isometrically act on $(V, \mathcal{G}_{\mathrm{best}})$. But the high dimensional topology (unknown to Hopf) tells us that the space Diff V is too vast, soft and unruly to be contained in something nice and cosy like the desired $\mathcal{G}_{\mathrm{best}}$. (Diff is governed by Waldhausen K-theory bringing lots of homotopy to Diff which hardly can be accounted for by a rigid geometry. And prior to Waldhausen our dream was shattered by Milnor's spheres ruling out smooth canonical deformations of general metrics on the spheres S^n, $n \geq 6$, to the standard ones.)

Besides topology, there is a geometric reason why we cannot freely navigate in the rugged landscapes of spaces like Diff. To see the idea, let us look at the simpler problem of finding the "best" closed curve in a given free homotopy class of loops in a Riemannian manifold V. If $K(V) \leq 0$, for instance, the gradient flow (of the energy function on curves) happily terminates at a unique geodesic: this is the best we could hope for. But suppose $\pi_1(V)$ is computationally complicated, i.e. the word problem cannot be solved by a fast algorithm, say, unsolvable by any algorithm at all. Our flow (discretized in an obvious way) is a particular algorithm, we know it must *badly* fail, and the only way for it to fail is to get lost and confused in deep local minima of the energy. Thus our V must harbor lots and lots of locally minimal geodesics in each homotopy class, in particular, infinity of contractible closed geodesics, disrupting the route from topology to simple geometry.

Well, one may say, let us *assume* $\pi_1(V) = 0$. But what the hell does it mean "assume"? Given a V, presented in any conceivable *geometric* form (remember, we are geometers here, not shape-blind topologists), there is no way to check if $\pi_1(V) = 0$ since this property is not algorithmically verifiable. Consequently, there are innocuously looking metrics on such manifolds as S^n for $n \geq 5$, teeming with short closed curves which no human being can contract in a given stretch of time. (In fact, a predominant majority of metrics are like this for $n \geq 5$.) A similar picture arises for higher dimensional (e.g. minimal) subvarieties (with extra complication for large dimension and codimension, even for simplest V such as flat tori, where the trickery of minimal subvarieties was disclosed by Blaine Lawson), and by the work of Alex Nabutovski the spaces like $\mathcal{G}(V)/\operatorname{Diff}(V)$ harbor the same kind of complexity rooted in the Gödel-Turing theorem.

Following Alex we (I speak for myself) are lead to the pessimistic conclusion that there is no chance for a distinguished g_{best} (or even $\mathcal{G}_{\mathrm{best}} \subset \mathcal{G}$) for $n \geq 5$ and that "natural" metrics, e.g. Einstein $\mathcal{G}$ with $\mathrm{Ri}(g) = \lambda g$ for $\lambda < 0$, must be chaotically scattered in the vastness of $\mathcal{G}$ with no meaningful link between geometry and topology. (This does not preclude, but rather predicts, the existence of such metrics, e.g. Einstein, on all V of dimension ≥ 5: the problem is there may be too many of them.)

On the optimistic side, we continue searching for g_{best} in special domains in $\mathcal{G}$, e.g. following Hamilton's Ricci flow (say, for Ricci ≥ 0 or $K \leq 0$) or stick to dimensions three (where Michael Anderson makes his theory) or four (where Taubs discovered certain softness in selfduality). Alternatively, we can enlarge (rather than limit) the category and look for extremal (possibly) singular varieties with only partially specified topology, i.e. with a prescribed value of a certain topological invariant, such as a characteristic number or the simplicial volume. For example, each (decent) topological space X is, rather canonically, accompanied by metric spaces homotopy equivalent to it, such as a suitably subdivided semisimplicial model of X which is an infinite dimensional simplicial complex, call it $X_\Delta \overset{\text{hom}}{\sim} X$, where the metric on X_Δ comes from a choice of a standard metric on each k-simplex, $k = 1, 2, \ldots$. This X_Δ is too large to please our eye, but it may contain some distinguished subvarieties, e.g. minimizing homology classes in $H_*(X_\Delta) = H_*(X)$, at least for certain spaces X (see 5 H_+ in [G-L-P] and references therein).

(f) Anisotropic metrics appear under a variety of names: "non-holonomic", "control", "sub-elliptic", "sub-Riemannian", "Carnot-Caratheodory", bearing the traces of their origin. They were extensively studied by analysts since Hörmander's work on hypo-elliptic operators but I do not know where and when they were promoted from technical devices to full membership in the metric community. Now, besides P.D.E., they help group theorists in the study of discrete nilpotent groups Γ since the *local* geometry of (especially self-similar) metrics on nilpotent Lie groups $G \supset \Gamma$ adequately reflects the *asymptotic* geometry of Γ. The local geometry of dist_H reduces, in turn, to that of the polarization $H \subset T(G)$ which is rather soft as far as low dimensional H-horizontal (i.e. tangent to H) subvarieties in G are concerned as follows from the Nash implicit function theorem. This provides some information on dist_H-minimal subvarieties in G (alas, only the case of surfaces is understood) and allows us to evaluate the *Dehn* function of Γ (see [$\mathrm{Gro}_{\mathrm{CC}}$], [$\mathrm{Gro}_{\mathrm{AI}}$] and references therein). Also observe that (however meager) results and problems in the anisotropic geometry may serve as model for other soft spaces (of solutions of P.D.E. as H-horizontality for H).

(g) It is helpful to think of a (metric) measure space (X, μ) as a high dimensional configuration space of a physical system (which is, indeed, often the case). Here $f : X \to Y$ is an *observable* projecting X to a low dimensional "screen", our Y such as the space-time $\mathbb{R}^4$, for instance, where $f_*(\mu)$ is what we see on the screen. The concentration of $f_*(\mu)$, ubiquitous in the probability theory and statistical mechanics, was brought to geometry (starting from Banach spaces) by Vitali Milman following the earlier work by Paul Levy. The Levy-Milman concentration phenomenon has been observed for a wide class of examples, where, besides the mere concentration, one is concerned with large deviations and fluctuations. (See [Mi], [G-L-P] and references herein.) Unfortunately, our definition of concentrated spaces does not capture large deviations (which are more fundamental than fluctuations). Possibly, this can be helped by somehow enriching the structure. Besides, one can proceed by allowing variable measures (states) μ as in the cube $\{0,1\}^N$ with the product μ_p of N-copies of the $(p, 1-p)$-measure on $\{0,1\}$ and in Gibbs measures parametrized by temperature. Then, inspired by physics, one wonders what should be general objects responsible for concentration in quantum statistical mechanics (where the concentration is limited by the Heisenberg principle) and, finally, one may turn to non-metric structures which may come in probability and geometry along with concentration.

§ IV. Life without metric.

It is (too) easy to concoct invariants of a metric space V, e.g. by looking at the ranges of Cartesian powers of the distance function mapping V^k to $\mathbb{R}^{\frac{k(k-1)}{2}}$ for

$$d^k : (v_1, \dots, v_i, \dots, v_k) \mapsto \{\mathrm{dist}(v_i, v_j)\}_{1 \le i < j \le k}\,.$$

(The *diameter* of V appears, for example, as the maximal segment in the image of d^k in $\mathbb{R}^1_+$ for $k = 2$.) Then, there are various (often positive) "energy functions" E on spaces of subvarieties W in X and maps $f : V \to V_0$ such as $W \mapsto \mathrm{Vol}_m W$ for $m = \dim W$, and $\int \|df\|^2 dv$ for Riemannian V and V_0. Each E generates invariants of V, for instance, infima of E on given classes of W's or f's, (e.g. $\inf \mathrm{Vol}_m W$ for $[W] = h \in H_m(V)$) or, more generally, the full *Morse landscape* of E including the spectrum of the critical values of E (e.g. the spectrum of the Laplace operator on functions and form on a Riemannian V). These invariants I provide us with the raw material for asking questions and making conjectures: what are possible values of $I = I(E)$ and relations between I's for different E's? What are spaces V with a given behaviour of an I?, etc. But spaces without metrics become rather slippery, hard to grasp and assess. Just look at a foliation or dynamical system F on a manifold V. The essential invariants (entropy, asymptotics of periodic orbits) often change discontinuously under deformations (perturbations) of the structure and are hard to evaluate, even approximately, for a given F. And it is not easy at all to come up with new numerical invariants making sense for all objects in the category. This is due to the fact that the (local) group of (approximate) automorphisms of such a structure (at a point $v \in V$) is potentially *non-compact.* Consequently, the action of $\mathrm{Diff}(V)$ on the space $\mathcal{F}$ of our structures F on V may have non-trivial dynamics (e.g. non-compact isotropy groups $\mathrm{Is}_F = \mathrm{Aut}(V, F)$, $F \in \mathcal{F}$) making the quotient space $\mathcal{F}/\,\mathrm{Diff}\, V$, where our invariants are supposed to live, non-separable. (Intuitively, invariants should be independent of observers attached to different coordinate systems in V; if there are non-compactly many equivalent observers it becomes difficult to reconcile their views, as is in the special and general relativity, for example.)

Now we glance at a couple of *H-structures* for interesting non-compact subgroups $H \subset GL_n(\mathbb{R})$ (where the compact case of $H = O(n)$ corresponds to the Riemannian geometry).

$\mathbb{C}$-structures. These, customarily called *almost complex* structures J on V, are fields of $\mathbb{C}$-linear structures J_v in $T_v(V)$, $v \in V$. Such a J may be expressed by an anti-involution, also called $J : T(V) \to T(V)$ (corresponding to $\sqrt{-1}$), where the pertinent H is (non-compact!) $GL_m(\mathbb{C}) \subset GL_n(\mathbb{R})$ for $n = 2m$. Morphisms, called $\mathbb{C}$-*maps* $f : V_1 \to V_2$, where the differential $Df : T(V_1) \to T(V_2)$ is $\mathbb{C}$-linear (i.e. commute with J), are rare for (non-integrable) V_1 and V since the corresponding (elliptic) P.D.E.-system is overdetermined for $\dim_{\mathbb{R}} V_1 > 2$. So we stick to $\mathbb{C}$-*curves*, maps of Riemann surfaces $S \to V =$

(V, J), also called *J-curves* if this matters. We mark each S with a point $s \in S$ and then the totality of $\mathbb{C}$-curves makes a huge space $\mathcal{S} = \mathcal{S}(V, J)$ foliated into surfaces, where each leaf in $\mathcal{S}$ is represented by a fixed $S \to V$ with variable marking $s \in S$.

What is the (possible) global geometry (e.g. dynamics) of $\mathcal{S}$ and how can it be read from J ?

Start with the subfoliation $\mathcal{C} \subset \mathcal{S}$ of *closed* leaves in $\mathcal{S}$ corresponding to closed (Riemann) surfaces $S \to V$, and try to mimic the geometry of curves in complex algebraic (first of all, projective) varieties V. This $\mathcal{C}$ is filtered by the *degrees* d of $S \subset V$ (playing the role of periods of closed orbits, say for actions of $\mathbb{Z}$ on some space) where curves of degree d may degenerate to (several) curves of lower degree thus compactifying each (moduli) space $\mathcal{C}_d$ by low dimensional strata built of $\mathcal{C}_i$ with $i < d$. Conversely, one can often *fuse* lower degree curves to higher degree d by deforming their (reducible) unions to irreducible $S \in \mathcal{C}_d$.

What happens to this web of algebraic curves of degree d when we slightly perturbe the underlying J_0 ?

The answer depends on the *virtual* dimension of the space $\mathcal{C}_d$, i.e. the Fredholm index of the elliptic system defining $S \in \mathcal{C}_d$. For example, the curves in abelian varieties $\mathbb{C}^m/\Lambda$ are unstable under (even integrable) deformations of J_0, but the curves in certain V's (with sufficiently ample anti-canonical bundles) such as $\mathbb{C}P^m$, remain essentially intact under small (and large as we shall see below) deformations of J_0 (yet the shape and position of $\mathcal{C}_d$ in $\mathcal{S}$ may be, a priori, greatly distorted by ε-deformations of J_0 for d large compared to ε^{-1}). As we follow a deformation J_t moving it further away from the original J_0, a curve $S_0 \subset V$, persistent for small t, may eventually perish by blowing up to something non-compact as t reaches some critical value t_c. What is needed to keep S alive (as a *closed* J_t-curve) is an a priori bound on the area of $S \to (V, J = J_t)$ (measured with some background Riemannian metric g in V, where a specific choice of g is not important as we deal here with compact V). Such a bound is guaranteed by the following *tameness* assumption of J which limits the area of (even approximately) J-holomorphic curves S by their topology, namely by $[S] \in H_2(V)$.

Call J *tamed* by a *closed* 2-form ω on V, if ω is positive on all J-curves in V (i.e. $\omega(\tau, J\tau) > 0$ for all non-zero vectors $\tau \in T(V)$). If so, J is *tame with respect to the cohomology class $h = [\omega] \in H^2(V)$*: the area of each closed oriented "approximately J-holomorphic" curve $S \subset V$ is bounded in terms of the value $h[S] \in \mathbb{R}$. To make it precise, denote by $S_\varepsilon \subset S$ the set of points $s \in S \subset V$, where the plane $T_s(S) \subset T_S(V)$ is ε-close to a $\mathbb{C}$-line (for a fixed background metric). Then "tame" means the existence of $\varepsilon, \delta, C > 0$, such that the inequality $\operatorname{area} S_\varepsilon \geq (1 - \delta) \operatorname{area} S$ implies $h(S) \geq C^{-1} \operatorname{area} S$ for all closed oriented surfaces $S \subset V$. Clearly, "ω-tame" $\Rightarrow$ "h-tame" but the converse ($\forall h \exists \omega \ldots$) remains questionable. (One may try the Hahn-Banach theorem, especially for $\dim V = 4$.)

If (V, J_0) is a Kähler (e.g. algebraic) manifold, then J_0 is tamed by the (symplectic) Kähler form ω and as far as J_t remains $[w]$-tame we have decent moduli spaces of J_t-holomorphic curves in $(V,\ J_\varepsilon)$ (where $J = J_t$ may be quite far from J_0, e.g. $J = A\,J_0$, for an arbitrary symplectic automorphism $T(V) \to T(V)$). For example, if we start with the standard $(\mathbb{C}P^m,\ J_0)$ and $J = J_1$ is joined with J_0 by a homotopy of $[\omega_0]$-tame structure J_t (for the standard symplectic 2-form ω_0 on $\mathbb{C}P^m$), then $(\mathbb{C}P^m,\ J_1)$ admits a rational J_1-curve S of degree 1 through each pair of points, where, moreover, S is unique for $m = 2$. (This remains true for all ω-tame structures on $\mathbb{C}P^2$, with *no* a priori assumption $\omega = \omega_0$, by the work of Taubs and Donaldson.) But what happens to closed $\mathbb{C}$-curves at the first moment t_c when J_t loses tameness? What kind of subfoliation $\mathcal{S}_d \subset \mathcal{S}$ is formed by the limits of $S \in \mathcal{C}_d$ for $t \to t_c$? It seems, at least for $\dim V = 4$ (e.g. for $\mathbb{C}P^2$ and $S^2 \times S^2$), that most of the closed $\mathbb{C}$-curves blow up simultaneously forming a regular (foliated-like) structure in V. (This is reminiscent of how Kleinian groups degenerate remaining discrete and beautiful at the verge of distinction.)

Are there non-tame (V, J) *with rich moduli spaces of closed (especially rational) curves, say having such a curve passing through each pair of points in* V*?* (If a 4-dimensional (V, J) has many J-curves, it is tame by an easy argument. On the other hand the majority of higher dimensional (V, J) contain isolated pockets of J-curves with rather shapeless and useless C_d like closed geodesics in (most) Riemannian manifolds lost in accidental wells of energy.)

Turning to non-closed $\mathbb{C}$-curves we find a prerequisite for the Nevanlinna theory as they share (the principle symbol of) $\overline{\partial}$ with ordinary holomorphic functions and maps. (This is also crucial for the study of closed $\mathbb{C}$-curves.) For example, we can define *hyperbolic* (V, J) which receive *no* non-constant J-maps $\mathbb{C} \to V$ and these V (we assume compactness) carry a non-degenerate *Kobayashi metric*, i.e. the supremal metric for which the $\mathbb{C}$-maps $H^2 \to V$ are 1-Lipschitz. This hyperbolicity has a point in common with tameness: the space of $\mathbb{C}$-maps $f : S^2 \to V$ with $\overline{\partial} f$ ranging in a compact set is compact for hyperbolic V. (This is also implied by the "tame" bound area $S \leq \mathrm{const}([S])$ for approximately J-holomorphic spheres in V, provided there is no J-holomorphic spheres in V.) Consequently, for each $\mathbb{C}$-structure on $W = V \times S^2$ compatible with J on the fibers $V \times s,\ s \in S^2$, there is a rational (i.e. spherical) $\mathbb{C}$-curve in W passing through a given point $w \in W$ that *contractibly* projects to W. (This remains valid for irrational curves S if the Teichmuller space of S is incorporated into W.) Another link between "tame" and "hyperbolic" is expressed by the following (easy to prove) topological criterion for hyperbolicity. Let $\widetilde{V}$ be a Galois covering of V and $\tilde{\ell}$ be a 1-form with sublinear growth ($\|\tilde{\ell}(\tilde{v})\| / \mathrm{dist}(\tilde{v}, \tilde{v}_0) \to 0$ for $\tilde{v} \to \infty$), and with invariant (under the deck transformation group) differential $\tilde{w} = d\tilde{\ell}$. *If J is tamed by the corresponding class $[w] \in H_2(V)$, then (V, J) is hyperbolic.* (For example, if J is tame, $\pi_2(V) = 0$ and $\pi_2(V)$ is hyperbolic, then (V, J) is hyperbolic.)

Are there further topological criteria for (non)-hyperbolicity (where $\pi_1(V)$

is not so large)? For example, are there ω-tame hyperbolic structures on the 4-torus? (Parabolic curves, i.e. $\mathbb{C}$-maps into such torus (T^4, w, J) could help us to study ω, where the ultimate goal is to show that every symplectic structure ω on T^4 is isomorphic to the standard one as is known for $\mathbb{C}P^2$, for instance by the work of Taubs and Donaldson.)

What are essential metric properties of $(V, \mathrm{dist}_{\mathrm{Kob}})$ *for hyperbolic* V*?* Nothing is known about it. *How much can one deform a hyperbolic* J *keeping hyperbolicity?* It is clear (by Brody's argument) that small perturbations do not hurt. Take, for instance, a (necessarily singular) J-curve S_0 in an ω-tame $(\mathbb{C}P^2,\ J)$, where the fundamental group of the complement is hyperbolic (e.g. free noncyclic which is the case for S_0 consisting of $d+1$ rational curves of degree 1 where exactly d of them meet at a single point and $d \geq 3$). Then every non-constant J-map $\mathbb{C} \to (\mathbb{C}P^2,\ J)$ meets S_0 and if we slightly deform S_0, the complement of the resulting complement $\mathbb{C}P^2 \backslash S_\varepsilon$ remains hyperbolic, provided each irreducible component of S_0 minus the remaining components were hyperbolic (as happens if the above $d+1$ lines are augmented by another one meeting each at a single point). Thus, for every $d \geq 5$, there is a non-*empty* open subset $\mathcal{H}_d \subset C_d$ of smooth J-curves S in $(\mathbb{C}P^2,\ J)$ of degree d where every non-constant J-map $\mathbb{C} \to \mathbb{C}P^2$ meets S. Probably, this $\mathcal{H}_d$ is dense as well as open for large d possibly, depending on J, (where the case of curves with many, depending on J, irreducible components seems within reach). Similar observation can be made for other compact V and differences of these, e.g. for those associated to tori, but nowhere one comes close to what is known in the classical algebraic case. Here is another kind of (test) question with no classical counterpart: *Given a hyperbolic* (V, J_0)*, when and how can one modify* J_0 *inside a (arbitrarily) small neighbourhood* $U \subset V$*, such that the resulting* (V, J_1) *admits a parabolic (or even rational) curve through each point* $v \in V$ *?*

If (V, J) is hyperbolic then the space $\mathcal{H}$ of J-maps $H^2 \to V$ is *compact* and projects *onto* $\mathcal{S}$ with circle fibers. The group $G = PSL_2\mathbb{R} = \mathrm{Iso}\, H^2$ naturally acts on $\mathcal{H}$ and periodic (i.e. compact) orbits correspond to closed J-curves $S \subset V$, where genus $(S) \geq \mathrm{const}(\mathrm{Area}\, S)$. If (V, J) is algebraic, then periodic orbits are dense in $\mathcal{H}$; besides, there are many finite dimensional invariant subsets in $\mathcal{H}$ corresponding to (solutions of) algebraic O.D.E. over V. *Are finite dimensional invariant subsets dense in* $\mathcal{H}$ *for all (tame)* V *? When does* V *contain a metrically complete* J*-curve* $S \subset V$*, preferably with locally bounded geometry? What are (if any) invariant measures in* $\mathcal{H}$*? Can one extract symplectic invariants of* ω *out of* $\mathcal{H}$ *for* ω*-tame* J *in the absence of closed* J*-curves?* Take, for example an algebraic (V, J_0) with a nice foliation, e.g. a flat connection over a Riemann surface or the standard foliation on a Hilbert modular surface. *How does such a foliation fare under tame homotopies* J_t *?*

If (V, J) is non-hyperbolic then the main issue is to understand the space $\mathcal{P} \subset \mathcal{S}$ of parabolic leaves, or equivalently, the space $\mathcal{P}'$ of non-constant J-maps $\mathbb{C} \to V$ with the action of $G = \mathrm{Aff}\,\mathbb{C}$. One knows, for the standard $\mathbb{C}P^n$, that rational maps are dense in $\mathcal{P}'$ (and hence in $\mathcal{H}$) by the Runge theorem. This, probably, is true for all tame (V, J) with "sufficiently many" rational curves (but this seems unknown even for rationally connected *algebraic* V). Let, for instance, $V = (\mathbb{C}P^n, J)$ where J is tamed by the standard ω. Then, by fusing (a sequence of) rational curves, one can obtain a parabolic one containing a given countable subset in V. Most likely, one can prove the Runge theorem(s) for V in this way following Donaldson's approach to Runge for Yang-Mills.

Denote by $\mathcal{B} \subset \mathcal{P}'$ the space of *non-constant* 1-Lipschitz $\mathbb{C}$-maps $\mathbb{C} \to V$ and observe that the closure of every G-orbit in $\mathcal{P}'$ meets $\mathcal{B}$ by the Bloch-Brody principle. If V contains (many) rational curves, then $\mathcal{B}$ is quite large, e.g. if $V = (\mathbb{C}P^n, J_{\text{stand}})$, then there is a map $f \in \mathcal{B}$ interpolating from an arbitrary δ-separated subset $\Delta \subset \mathbb{C}$; probably, this remains true for all J tamely homotopic to the standard one as well as for more general "rationally connected" V. Conversely, (a strengthened version of) this interpolation property is likely to imply the existence of (many) rational curves in V.

Apart from rational curves, one can sometimes produce $\mathbb{C}$-maps $\mathbb{C} \to V$ by prescribing some asymptotic boundary conditions, e.g. by proving non-compactness of the space of J-disks with boundaries on a non-compact family of Lagrangian subvarieties. This works for many J on the *standard* symplectic $\mathbb{R}^{2m}$ and yields, for instance, parabolic curves for J on tori tamed by the standard ω. *How large is $\mathcal{B}$ for these tori?* Hopefully, the mean (ordinary?) dimension of $\mathcal{B}$ (for the action of $\mathbb{C} \subset G$ on $\mathcal{B}$) is finite and the natural map from $\mathcal{B}$ to the projectivized $H_2(V)$ is non-ambiguous and somehow represents the (homology) class of $\mathbb{C}P^{m-1}$ corresponding to $\mathbb{C}$-lines in $H_1(V) = \mathbb{C}^m = \mathbb{R}^{2m}$.

Pseudo-Riemannian manifolds. Given an H-structure g on a manifold V for a non-compact group $H \subset GL_n(\mathbb{R})$, $n = \dim V$, one may rigidify (V, g) by reducing H to a maximal compact subgroup $K \subset H$, i.e. by considering a Riemannian metric g_K on V compatible with g. For example, if $g(= J)$ is a $\mathbb{C}$-structure, then $K = U(m) \subset GL_m(\mathbb{C})$, $m = \frac{n}{2}$, and g_K is an Hermitian metric; if g is pseudo-Riemannian, i.e. a quadratic differential form on V of type (p, q) with $p + q = n$, then $K = O(p) \times O(q) \subset O(p, q)$ and g_K is a Riemannian metric such that there is a frame $\tau_1^+, \dots, \tau_p^+, \tau_1^-, \dots, \tau_q^-$ at each point, where both forms g and g_K become diagonal with $g_K(\tau_i^+, \tau_i^+) = g(\tau_i^+, \tau_i^+)$ and $g_K(\tau_j^-, \tau_j^-) = -g(\tau_j^-, \tau_j^-)$. *What properties (invariants) of g can be seen in an individual g_K and/or in the totality $\mathcal{G}_K = \mathcal{G}_K(g)$ of all g_K?* It may happen that two structures g and g' are virtually indistinguishable in these terms, namely when for each $\varepsilon > 0$, there exist g_K and g'_K which are $(1+\varepsilon)$-bi-Lipschitz equivalent. (The Diff-orbit of $\mathcal{G}_K$ might be C^0-dense in the space of all Riemannian metrics under the worst scenario.) There are few known cases where $\mathcal{G}_K$ tells you something useful about g. An exceptionally pleasant example is given by *conformal* structures g with $\mathcal{G}_K(g)$ telling you everything about g (i.e. the C^0-closures of the Diff-orbits of $\mathcal{G}_K$ and $\mathcal{G}'_K$ are essentially disjoint unless g and g' are isomorphic; furthermore the action of Diff V on the space of conformal structures is proper apart from the standard conformal S^n). Next, if H preserves an exterior r-form on $\mathbb{R}^n$, and the corresponding form $\Omega = \Omega(g)$ on V is closed, then g_K is bounded from below by volumes of r-cycles $C \subset V$ with $\Omega(C) \neq 0$ and, at least, $\mathrm{Diff}\,\mathcal{G}_K$ is not dense. Finally, for certain symplectic structures $g = \omega$, the minimal g_K-areas of some 2-cycles C (realizable by $\mathbb{C}$-curves) equal $\omega(C)$ for all g_K, thus limiting geometry of (V, g_K) in a more significant way.

Having failed with (robust) Lipschitz geometry of g_K one resorts to curvature and looks at the subsets $\mathcal{G}_K(\kappa) \subset \mathcal{G}_K$ of adapted metrics $g_K = g_K(g)$ with some (norm of) curvature bounded by $\kappa \in \mathbb{R}$. Here the mere fact of $\mathcal{G}_K(\kappa)$ being empty for a given κ, gives one a non-trivial complexity bound on g. For example, one may study the infimal $\kappa \in \mathbb{R}_+$, such that g admits g_K with the sectional curvatures between $-\kappa$ and κ (with normalized volume if $H \not\subset SL_n\mathbb{R}$). And again one is tempted to search for the "best" g_K adapted to g with non-zero expectation for "mildly non-compact" H inspired by conformal metrics of constant scalar curvature delivered by the Schoen-Aubin solution to the Yamabe problem.

Now we turn to the case at hand, pseudo-Riemannian g of type (p, q) on a (typically compact) manifold V having closer kinship to Riemannian geometry than general g. *What are (most general) morphisms in the pseudo-Riemannian category comparable to* "Lipschitz" *for Riemannian manifolds?*

Since we can compare metrics on a fixed manifold by $g \leq g'$ for $g - g'$ being positive semidefinite, we may speak of $(+)$-*long* maps $f : V \to V'$ where $f^*(g') \geq g$. However, unlike the Riemannian case where short (rather than long) maps are useful for all V and V', this makes sense only for (p, q) not being too small compared to (p', q'): if $p' > p$ then every isometric immersion $f_0 : V \to V'$ can be a little C^0-perturbed to some f with $f^*(g')$ being as large as you want (actually equal a given $g_1 > f_0^*(g')$ homotopic to g by the Nash-Kuiper argument), and if also $q' > q$, everything becomes soft and one gets all metric on V (homotopic to g) by C^1-immersions arbitrarily C^0-close to f_0. Thus we stick to $p = p'$ and start with *positive slices* in V, i.e. immersed p-dimensional $W \subset V$ with $g|W > 0$. If W is connected with *non-empty* boundary, we set $R(W, w) = \mathrm{dist}(w, \partial W)$ for the induced Riemannian metric ($\sup\limits_w R(w, W)$ is called "in-radius" of W), and define $R_+(V, v)$ as the supremum of $R(v, W)$ for all positive slices through v. (Connected *Riemannian* manifolds with *non-empty* boundaries have $R_+ = R < \infty$ at all their points while closed manifolds have $R_+ < \infty$ if and only if $\pi_1 < \infty$. But even for $\pi_1 = \{e\}$ there is no *effective* bound on R_+ due to non-decidability of $\pi_1 \overset{?}{=} \{e\}$. Moreover, there are rather small metrics on S^3 with almost negative curvature and thus with arbitrarily large R_+). Next, we dualize by taking $(+)$-long maps f from (V, v) to Riemannian (W, w) of dimension p and denote by $\underline{R}_+(V, v)$ the infimum of $R_+(W, w)$ over all possible (W, w) and f, where, clearly, $\underline{R}_+ \geq R_+$.

A $(+)$-long map $V \to W$ is necessarily a *negative* submersion, i.e. with g-negative fibers. Conversely, given a negative submersion f (proper, if V is non-compact) of V to a smooth manifold W, one can find a Riemannian metric on W making f long. (There is a unique supremal Finsler metric on W making f long, which can be minorized by a Riemannian one.) Thus, for example, if a compact (possibly with a boundary) V admits a negative submersion into a connected manifold with finite π_1 (or with non-empty boundary), then $R_+(V) < \infty$.

Manifolds with $R_+(V, v) \leq \text{const} < \infty$ for all $v \in V$ are kind of hyperbolic in the $(+)$-directions (e.g. this condition is C^0-stable). If $p = 1$ and V is closed, then it always has a circular positive slice and $R_+(V, v) = \infty$ for all $v \in V$, but if $p \geq 2$, then every $g = g_0$ admits a deformation g_t with $R_+ \to 0$ for $t \to \infty$: take a generic (and thus non-integrable) p-plain field $S_+ \subset T(V)$ with $g|S > 0$ and make $g_t = g + tg_-$ where g_- is the negative part of g for the normal splitting $T(V) = S_+ \oplus S_+^\perp$. (This works even locally and shows that the majority of g have R_+ small in some regions of V and infinite in other regions.)

It seems by far more restrictive to require that both R_+ and R_- (i.e. R_+ for $-V$) are bounded on V, where a sufficient condition for compact V is the existence of a negative submersion $V \to W_+$ as well as a positive submersion $V \to W_-$, where W_+ and W_- have $\pi_1 < \infty$ or non-empty boundaries. This can be slightly generalized by allowing somewhat more general pairs of $\pm$-foliations with uniformly compact fibers (e.g. coming from submersion to simply connected or to bounded *orbifolds*) and sometimes one submersion suffices. For example, start with a negative submersion $f : V \to W_+$ where W_+ is simply connected (or bounded) and the (negative) fibers of f are also simply connected (or bounded). Deform the original g on V to g_1 agreeing with g on the fibers while being very positive normally to the fibers. Then all negative slices in (V, g_1) keep C^1-close to the fibers of f and so $R_-(V, g_1)$ is bounded as well as R_+. (Probably, there are more sophisticated, say closed manifolds V, where the bound on $R_\pm(V, v)$ comes from different sources at different $v \in V$.)

Despite some hyperbolic features, $\pm$ bounded pseudo-Riemannian metrics on closed manifolds are reminiscent of positive curvature, e.g. they are accompanied in known examples by *closed* positive or negative slices with $\pi_1 < \infty$ (are these inevitable?) and seem hard to make on aspherical V.

Besides taming g by $\pm$ foliations, one may try (pairs of) differential forms on V, where a closed p-form ω is said to (strictly) $(+)$-*tame* g if it is (strictly) nonsingular on the positive slices. For example, if V is metrically split, $V = V_+ \times V_-$, the pull-back ω_+ of the volume form of V_+ strictly $(+)$-tames g; similarly ω_- strictly $(-)$-tames g, while $\omega_+ + \omega_-$ is $\pm$ taming, albeit non-strictly. *How much do closed positive slices persist under (strictly)* $(+)$*-tame homotopies of* g*? In particular, what happens to area maximizing closed positive 2-slices in* $(2, q)$*-manifolds under strictly* $(+)$*-tame homotopies?*

The tangent bundle to the space $\mathcal{S}_-$ of negative slices in V carries a natural (positive!) L_r-norm (we use r as p is occupied) since g is positive normally to $S \subset V$ for all $S \in \mathcal{S}_-$ (as well as for all $S_{p',q} \subset V$ with $0 \leq p' < p$). But the associated (path) metric in $\mathcal{S}_-$ may, a priori, degenerate and even become everywhere zero. Yet, there are some positive signs.

Let V metrically split $V = [0, 1] \times V_-$ for V_- closed. Then the L_1-distance between $0\times V_-$ and $1\times V_-$ equals the volume of V and hence >0. Consequently all L_r-distances are >0.

Let $V = V_+ \times V_-$ with closed V_- and arbitrary Riemannian V_+. Then $\mathrm{dist}_{L_\infty}(v_+ \times V_-,\ v'_+ \times V_-) = d = \mathrm{dist}_{V_+}(v_+,\ v'_+)$, since the projection $V \to V_-$ together with a 1-Lipschitz retraction

$$V_+ \to [0,d] = [v_+,\ v'_+]$$

give us a $(+)$-short (i.e. $(-)$-long) map $V \to [0,d]\times V_-$ and the above applies (only to L_∞ not necessarily to other L_r).

Let V fiber over a compact p-manifold with closed negative fibers. Then the L_∞-distance is >0 between every two distinct fibers, essentially by the same argument (also yielding positivity of L_∞-distances between general closed negative slices isotopic to the fibers).

How much does the metric geometry of $(\mathcal{S}_-,\ \mathrm{dist}_{L_\infty})$ (and of $\mathcal{S}_+$) *capture the structure of* g*? How degenerate* dist_{L_∞} *can be for general* (V,g)*?* (This dist_{L_∞} is vaguely similar to Hofer's metric in the space of Lagrangian slices in symplectic V, which also suggests vanishing of our dist_{L_r}, $r<\infty$, for most V.)

As we mentioned earlier, a general complication in the study of H-structures g with non-compact $H \subset GL_n\mathbb{R}$ is a possible *non-stability* (or recurrency) of g due to certain unboundedness of the set of diffeomorphisms of V moving g (or a small perturbation of g) close to g. The simplest manifestation of that is non-compactness of the automorphisms (isometry) group of (V,g) which may have different nature for different structures g. For example, non-compactness of the conformal transformations f of S^n is seen in the graphs $\Gamma_f \subset S^n \times S^n$ as degeneration of these to unions of two fibers $(s_1 \times S^n)\cup(S^n \times s_2)$ with an uniform bound on $\mathrm{Vol}\,\Gamma_f$ for $f\to\infty$. On the other hand, graphs of isometrics f of (p,q)-manifolds V are represented by (totally geodesic) *isotropic* (where the metric vanishes) n-manifolds $\Gamma_f \subset V \times -V$ and their volumes (as well as in-radii, both measured with respect to some background Riemannian metric in $V\times V$) go to infinity for $f\to\infty$, while their local geometry remains bounded (unlike the conformal case). With this in mind, we call g *stable*, if it admits a C^0-neighbourhood $\mathcal{U}$ in the space $\mathcal{G}$ of all g's, such that the graphs of isometrics $f:(V,\ g')\to(V,\ g'')$ with $g',\ g''\in\mathcal{U}$ have $\mathrm{Vol}_n(\Gamma_f)\le \mathrm{const}<\infty$ (where the background metric is not essential as we assume V compact).

Example. Start with a metrically split $V = V_+\times V_-$. The isometrics here are essentially the same as those of the Riemannian manifold $V^+ = V_+\times(-V_-)$, since $V\times(-V) = V_+\times V_-\times(-V_+)\times(-V_-) = V_+\times(-V_-)\times(-V_+)\times V_- = V^+\times(-V^+)$, and so $V\times(-V)$ and $V^+\times(-V^+)$ share the same isotropic submanifolds. If V is closed simply connected (or f does not mix up $\pi_1(V_+)$ and $\pi_1(V_-)$ too much) one sees, by looking at (local) isometries of V^+, that all isotropic submanifolds in $V\times(-V)$ have bounded in-radii (as well as volumes, if they are closed) and so $\mathrm{Iso}(V,g)$ is compact. One sees equally well that

$g = g_+ \oplus g_-$ is stable, and, moreover, one gets a good control over the stability domain $\mathcal{U}$ of g. Namely, take $\lambda > 0$ and let $\mathcal{U}_\lambda = \mathcal{U}_\lambda(g)$ consist of those g', where the fibers $V_+ \times v_-$ and $v_+ \times V_-$ are g'-positive and g'-negative correspondingly and the projections of these fibers to (V_+, g_+) and (V_-, g_-) are λ-bi-Lipschitz for g'-restricted to these fibers. Then (at least for $\pi_1(V) = 0$) the graphs of diffeomorphism $f : V \to V$ with $\mathcal{U}_\lambda \cap f(\mathcal{U}_\lambda) \neq \emptyset$ are uniformly bi-Lipschitz to V^+, and so all $g' \in \bigcup\limits_{\lambda<\infty} \mathcal{U}_\lambda$ are stable.

What are most general stable g? Are simply connected manifolds of type $(1, q)$ always stable? (Of course, *generic* g are stable, but we are concerned with exceptional (V, g), e.g. with non-compact group $\mathrm{Iso}(V, g)$.)

The above motivates the idea of *iso-stability* for V of type (p, q) with $p = q$ (e.g. $V = V_0 \oplus -V_0$) limiting the size of ϵ-isotropic submanifolds in V. This is enhanced in the presence of 0-*taming* p-forms ω on V, which do not vanish on the isotropic p-planes in $T(V)$. For example, if $V \times (-V)$ is 0-tame (with $\deg \omega = n$ on this occasion), then V is stable with respect to the diffeomorphisms with the graphs homologous to the diagonal in $V \times V$, as is the case for the above $g \in \mathcal{U}_\lambda$ and for (V, g) tamed by $\pm$-foliations with p and q-volume preserving (or just uniformly bounded) holonomies.

Remarks and references.

(a) Closed $\mathbb{C}$-curves in tame manifolds exhibit a well organized structure with intricate interaction between moduli spaces $\mathcal{C}_d$ for different d and regular asymptotics for $d \to \infty$: quantum multiplication, mirror symmetry etc (see [McD-Sal]). This also applies to non-closed curves with prescribed Fredholm boundary (or asymptotic) condition, e.g. J-maps of Riemann surfaces with boundaries, $(S, \partial S) \to (V, W)$ for a given *totally real* $W \subset V$, where everything goes as in the boundary free case (including Kobayashi metric, Bloch-Brody etc). Less obvious conditions come up in the study of fixed points of Hamiltonian transformations and related problems: Floer homology, A-categories, contact homology of Eliashberg and Hofer. But it is unclear (only to me?) what is the most general Fredholm condition in the $\mathbb{C}$-geometry.

(b) The questions concerning unbounded $\mathbb{C}$-curves, which parallel (Nevanlinna kind) complex analysis rather than algebraic geometry, remain as widely open as when I collected them for (then expected) continuation of [$\mathrm{Gro}_{\mathrm{PCMD}}$]. *Do the spaces $\mathcal{S}, \mathcal{H}, \mathcal{P}, \mathcal{B}$ possess geometric structure comparable to (and compatible with) what we see in $\mathcal{C} = \bigcup_d \mathcal{C}_d$? What is the right language to describe such a structure (if it exists at all)?*

Even in the classical case of algebraic V boasting of lots and lots of deep difficult theorems, there is no hint of the global picture in sight, not even a conjectural one (see [McQ] for the latest in the field).

(c) Hyperbolicity of (V, J) can be sometimes derived from negativity of a suitable curvature of a (Riemann or Finsler) metric adapted to J, either on

V itself or on some jet space of $\mathbb{C}$-curves in V. A most general semi-local hyperbolicity criterion is expressed by the *linear isoperimetric inequality* in $\mathbb{C}$-curves $S \subset V$. If such an inequality holds true on relatively small J-surfaces $S \subset V$, then it propagates to all S in the same way as the real hyperbolicity does (see [$\mathrm{Gro}_{\mathrm{HG}}$]) which makes it, in principle, verifiable for compact V. (The linear inequality seems to *follow* from hyperbolicity by a rather routine argument, but I failed to carry it through. Possibly, one should limit oneself to closed V and integrability of J may be also helpful.)

It would be amusing to find a sufficiently general (positive) curvature condition for the existence of many rational curves in (V,J), encompassing complex hypersurfaces V with $\deg V$ (much) smaller than $\dim V$, for example, and allowing singular spaces in the spirit of Alexandrov's $K \geq 0$. Conversely, in the presence of many closed (especially rational) curves, one expects extra local structures on V, e.g. taming forms ω (which easily come from closed curves for $\dim V = 4$, see [$\mathrm{Gro}_{\mathrm{PCMD}}$]).

(d) One can sometimes make foliations (or at least, laminations) out of parabolic curves in V as is done in [Ban] for J on tori tamed by the *standard* symplectic ω (whereas the original question aimed at eventually *proving* that ω is standard).

(e) *How much do we gain in global understanding of a compact* (V,J) *by assuming that the structure* J *is integrable (i.e. complex)?* It seems nothing at all: there is no single result concerning *all* compact complex manifolds. (If $\dim V = 4$, then the Kodaira classification tells us quite a bit, say for *even* $b_1 \geq 2$, especially if there are 4 elements in $H^1(V)$ with non-zero product yielding a *finite* morphism of V onto $\mathbb{C}^2/\Lambda$.) This suggests the presence of (unreachable?) pockets of (moduli spaces of) integrable J's with weird properties (like those produced by Taubs on 6-manifolds); but there is no general existence theorem for complex structures either (not even for open V's, compare p. 103 in [Gro]$_{\mathrm{PDR}}$) and even worse, no systematic way to produce them. So far, COMPACT COMPLEX MANIFOLDS have not stood to their fame.*

(f) $\mathbb{C}$-curves, defined by restricting their tangent planes to the subvariety $E_0 = \mathbb{C}P^m \subset Gr_2\mathbb{R}^{2m}$, owe their beauty and power to the *ellipticity* of E_0: there is a single plane $e \in E_0$ through each line in $\mathbb{R}^{2m}$. One can deform E_0 by keeping this condition thus arriving at generalized $\mathbb{C}$-structures where the resulting *E-curves* are similar to $\mathbb{C}$-curves, and where the picture is the clearest for $m = 2$ (see [$\mathrm{Gro}_{\mathrm{PCMD}}$] and references therein). In general, a field E of subsets $E_v \subset Gr_k T_v(V) = Gr_k\mathbb{R}^n$, $n = \dim V$, defines a class of k-dimensional *E-subvarieties* in V, said "directed by E" (e.g. $W \subset \mathbb{R}^n$ with Gauss image in E_0), which seem most intriguing under *ellipticity* assumptions on E. To formulate these, let $F_0 = F_0(E_0)$, $E_0 \subset Gr_k\mathbb{R}^n$, denote the space of pairs (e,h) for $e \in E_0$ and all hyperplanes $h \subset e$, and look at the tautological map $\pi_0 : F_0 \to Gr_{k-1}\mathbb{R}^n$. Ideally, F_0 is a smooth closed manifold and π is *stably* one-to-one and onto (i.e.

*Fedia Bogomolov suggested to look at manifolds appearing as spaces of leaves of foliations in pseudoconvex bounded domains in $\mathbb{C}^N$ with algebraic tangent bundles.

remains such under small perturbations of E_0) as in the $\mathbb{C}$-case. Such E_0 do not come cheaply: all known instances of them appear as (deformations of) Lie group orbits in $Gr_k\mathbb{R}^n$, e.g. in Harvey-Lawson calibrated geometries (where a most tantalizing $E_0 \subset Gr_3\mathbb{R}^7$ is associated to rational J-curves in S^6 for the standard G_2-invariant $\mathbb{C}$-structure J on S^6). One gains more examples by dropping "onto" thus arriving at overdetermined elliptic systems ("isotropic in pseudo-Riemannian", for instance) which need integrability in order to have solutions (as "special Lagrangian" of Harvey-Lawson). Also, one may allow E_0 and π_0 to have some singularities (similar to those present in Yang-Mills in a different setting), but in all cases one is stuck with two problems: *what are possible elliptic E_0, and what are (global and local) analytic properties (especially singularities) of the corresponding E-subvarieties W in $(V,\ E = \{E_v\}\,)$?* (If E is overdetermined, one looks at W's directed by a small neighbourhood $E_\varepsilon \supset E$.)

(g) The radii $\underline{R}_\pm$ are less useful than $R_\pm$ as they make sense only for rather special pseudo-Riemannian manifolds $V = (V, g)$. Yet $\underline{R}_\pm$ can be used for characterization of such V, e.g. the equality $\underline{R}_\pm = R_\pm < \infty$ seems to (almost?) distinguish metrically split manifolds. Also one can generalize $\underline{R}_\pm$ by considering submersions $V \to W_\pm$ being $\pm$ long *normally* to the fibers, where $1 \leq \dim W_+ \leq p$ and $1 \leq \dim W_- \leq q$ and where g is $\pm$ definite on the bundle of vectors normal to the fibers. Then the resulting radii $\underline{R}^\perp_\pm(V)$ satisfy $\underline{R}_\pm \geq \underline{R}^\perp_\pm \geq R_\pm$.

(h) The finiteness of $R_\pm(V)$ does not ensure stability of V for $\dim V \geq 4$ as simple (e.g. split) examples show but this seems to "limit instability to codimension two". *Can one go further with stronger radii-type invariants?*

(i) Besides the in-radius, there are other Riemannian invariants to gauge pseudo-Riemannian metrics such as the macroscopic dimension (see [$\mathrm{Gro}_{\mathrm{PCMD}}$]) of (complete) positive slices in V, or the maximal radius of a *Euclidean* p-ball in V. (The Euclidean metric on $\mathbb{R}^p$ dominates other g: there is a long map $\mathbb{R}^p \to (\mathbb{R}^p g)$ for every Riemannian metric g on $\mathbb{R}^p$. But one can go beyond $\mathbb{R}^p$ by admitting slices with non-trivial topology. Some V may contain lots of these, e.g. some V of large dimension support (p, q)-metrics g so that *every* Riemannian p-manifold admits an isometric immersion into (V, g).) Furthermore, one may look at homotopies and extendability of slices with controlled size thus getting extra invariants of V. Actually, the mere topology of the space of, say, closed positive slices can be immensely complicated encouraging us to seek conditions limiting this complexity (e.g. in the spirit of diagram groups, see [Gu-Sa]).

(j) If V is compact, one may distinguish *complete* slices for the metric induced from some Riemannian background h in V and then compare these induced Riemannian metrics with those induced from our g of type (p, q). (Besides completeness, h brings forth other classes of slices, e.g. those with some bounds on curvature.)

(k) The isometrics of (non-stable!) (V, g) have attractive geometry and dynamics (see [D'Amb-Gr] for an introduction and references) with many elementary questions remaining open, e.g. *does every isometry of the interior of V extend to the boundary?* (This comes from relativity, I guess.)

(l) Most of the current pseudo-Riemannian research is linked to general relativity focused on the Einstein equation (see[Be-Eh-Ea]).

§ V. Symbolization and randomization.

A common way to generate questions (not only) in geometry is to confront properties of objects specific to different categories: what is a possible *topology* (e.g. *homology*) of a manifold with a given type of *curvature?* How is the dynamics of the *geodesic* flow correlated with *topology* and/or *geometry?* How fast a *harmonic function* can decay on a *complete* manifold with a certain asymptotic *geometry*, e.g. curvature? How many *critical points* a *geometrically* defined energy may have on given space maps or subvarieties? What are possible *singularities* of the *exponential map* or of the *cut locus?* Does every (almost complex) *manifold* (of dim ≥ 6) support a *complex structure?* etc. These seduce us by simplicity and apparent naturality, sometimes leading to new ideas and structures (tangentially related to the original questions, as the Morse-Lusternik-Schnirelmann theory motivated by closed geodesics), but often the mirage of naturality lures us into featureless desert with no clear perspective where the solution, even if found, does not quench our thirst for structural mathematics. (Examples are left to the reader.)

Another approach consists in interbreeding (rather than intersecting) categories and ideas. This has a better chance for a successful outcome with questions following (rather than preceding) construction of new objects. Just look at how it works: symbolic dynamics, algebraic arithmetic and non-commutative geometry, quantum computers, differential topology, random graphs, p-adic analysis ... Now we want to continue with *symbolic geometry* and *random groups*.

Given a category of "spaces" X with *finite* Cartesian products, we consider formal *infinite* products $\mathcal{X} = \times_{i \in I} X_i$, where the index set has an additional (discrete) structure, e.g. being a graph or a discrete group Γ. In the latter case we assume that all X_i are the same, $\mathcal{X} = X^\Gamma$ consists of functions $\chi : \Gamma \to X$, and Γ naturally acts on $\mathcal{X}$. Nothing happens unless we start looking at morphisms $\Phi : \mathcal{X} \to \mathcal{Y}$ over a fixed Γ. Such a Φ is given by a finite subset $\Delta \subset \Gamma$ of cardinality d and a map $\varphi : X^\Delta = X^d \to Y$, i.e. a function $y = \varphi(x_1, \ldots, x_d)$, where $\Phi(\chi)(\gamma)$ is defined as the value of φ on the restriction of χ to the γ-translate $\gamma\Delta \subset \Gamma$ for all $\gamma \in \Gamma$. Thus we enrich the original category by making single variable (Γ-equivariant) functions $\Phi(\chi)$ out of functions $\varphi(x_1, \ldots, x_d)$ in several variables.

Take a particular category of X's, e.g. algebraic varieties, smooth symplectic or Riemannian manifolds, (smooth) dynamical systems, whatever you like, and start translating basic constructions, notions and questions into the "symbolic" language of $\mathcal{X}$'s. This is pursued in [$\mathrm{Gro}_{\mathrm{ESA}}$] and [$\mathrm{Gro}_{\mathrm{TID}}$] with an eye on continuous counterparts to $\mathcal{X}$, e.g. spaces of holomorphic maps $\mathbb{C} \to X$ for algebraic varieties X with a hope to make "algebraic" somehow reflected in such spaces. I have not gone far: a symbolic version of the Ax mapping theorem for amenable Γ (similar to the Garden of Eden in cellular automata) and a notion of *mean dimension* defined for all compact Γ-spaces $\mathcal{X}$ with amenable Γ (in the spirit of the topological entropy) recapturing $\dim_{\text{top}} X$ for $\mathcal{X} = X^\Gamma$ (applicable to spaces like $\mathcal{B}$ of J-maps $\mathbb{C} \to (X, J)$ for instance). That's about it. (The reader is most welcome to these $\mathcal{X}$; if anything, there is no lack of open questions; yet no guarantee they would lead to a new grand theory either.)

Randomization. Random lies at the very source of manifolds, at least in the smooth and the algebraic categories: general smooth manifolds V appear as pull-back of special submanifolds under *generic* (or random) smooth maps f between standard manifolds, e.g. zeros of generic functions $f : \mathbb{R}^N \to \mathbb{R}^{N-n}$ or (proper) generic maps from $\mathbb{R}^N$ to the canonical vector bundles W over Grassmann manifolds $\mathrm{Gr}_{N-n}\,\mathbb{R}^M$ (Thom construction), where V come as $f^{-1}(0)$ for the zero section $0 = \mathrm{Gr}_{N-n}\mathbb{R}^M \subset W$. (Other constructions in differential topology amount to little tinkering with V's created by genericity. Similarly, the bulk of algebraic manifolds comes from intersecting ample generic hypersurfaces in standard manifolds, e.g., in $\mathbb{C}P^N$, and the full list of known constructions of, say non-singular, algebraic varieties is dismally short.)

One may object by pointing out that every (combinatorial) manifold can be assembled out of simplices. Indeed, it is easy to make polyhedra, but no way to recognize *manifolds* among them (as eventually follows from undecidability of triviality for finitely presented groups). Here is another basic problem linked to "non-locality of topology". *How many triangulations a given space* X (e.g. *a smooth manifold, say the sphere* S^n) *may have?* Namely, let $t(X, N)$ denote the number of mutually combinatorially non-isomorphic triangulations of X into N simplices. Does this t grow *at most* exponentially in N?, i.e. whether $t(X, N) \leq \exp C_X N$. Notice that the number of *all* X built of N n-dimensional simplices grows super-exponentially, roughly as n^n, and the major difficulty for a given X comes from $\pi_1(X)$ and, possibly (but less likely), from $H_1(X)$, where the issue is to count the number of triangulated manifolds X with a fixed $\pi_1(X)$ or $H_1(X)$.

These questions (coming from physists working on quantization of gravity) have an (essentially equivalent) combinatorial counterpart (we stumbled upon with Alex Nabutovski): evaluate the number $t_L(N)$ of connected 3-valent (i.e. degree ≤ 3) graphs X with N edges, such that cycles of length $\leq L$ normally generate $\pi_1(X)$ (or, at least, generate $H_1(X)$)? *Is* $t_L(N)$ *at most exponential in* N *for a fixed* (say $= 10^{10}$) L? The questions look just great and no idea how to answer them.

Here is a somewhat similar but easier question: *what does a random group* (rather than a space) *look like?* As we shall see the answer is most satisfactory (at least for me): "nothing like we have ever seen before". (No big surprise though: typical objects are usually atypical.)

Random presentation of groups. Given a group F, e.g. the free group F_k with k generators, one may speak of random quotient groups $G = F/[R]$, where $R \subset F$ is a random subset with respect to some probability measure μ on 2^G and $[R]$ standing for the normal subgroup generated by R. The simplest way to make a μ is to choose weights $p(\gamma) \in [0,1]$ for all $\gamma \in \Gamma$ and take the product measure μ in 2^G, i.e. R is obtained by independent choices of $\gamma \in F$ with probabilities $p(\gamma)$. This is still too general; we specialize to $p(\gamma) = p(|\gamma|)$ where $|\gamma|$ denotes the word length of γ for a given, say finite, system of generators in F. A pretty such p is $p = p_\theta(\gamma) = (\mathrm{card}\{\gamma' \in F \mid |\gamma'| = |\gamma|\})^{-\theta}$, $\theta > 0$. If the "temperature" θ is close to zero, $p_0(\gamma)$ decays slowly and random R is so large that it normally generates all F making $G = \{e\}$ with probability one, provided F is infinite. For example, if $F = F_k$, this happens whenever $p(\gamma) \in \ell_2(F_k)$, i.e. $\sum\limits_\gamma p^2(\gamma) < \infty$. This is easy; but it is not so clear if G may be ever non-trivial for large θ. However, if $F = F_k$ (or a general non-elementary word hyperbolic group), one can show that G is infinite with positive probability for $\theta > \theta_{\mathrm{crit}}(F)$, and $\theta_{\mathrm{crit}}(F_k)$, probably, equals 2, i.e. $p_\theta(\gamma) \notin \ell_2 \Rightarrow \mathrm{card}(G) = \infty$ with non-zero probability (see [$\mathrm{Gro}_{\mathrm{AI}}$] for a slightly different $p(\gamma)$, where the critical 2 comes as the Euler characteristic of S^2 via the small cancellation theory).

Random groups G_θ look very different for different θ. It seems that G_{θ_1} cannot be embedded (even in a most generous sense of the word) to G_{θ_2} for $\theta_2 > \theta_1$ as the "density" of random G_θ decreases with "temperature". Furthermore, generic samples of G_θ for the same (large) θ are, probably, mutually non-isomorphic (not even quasi-isometric) with probability one, yet their "elementary invariants" are likely to be the same. It is clear for all $\theta < \infty$, that G_θ a.s. have no finite factor groups and they may satisfy Kazhdan's property T. (T is more probable for small θ where it is harder for G_θ to be infinite.)

Let us modify the above probability scheme by considering random homomorphism φ from a fixed group H to F with $G = F/[\varphi(H)]$. To be simple, let $H = \pi_1(\Delta)$ for a (directed) graph Δ and φ be given by random assignment of generators of F to each edge e in Δ, independently for all edges. Denote by $N(L)$ the number of (non-oriented) cycles in Δ of length $\leq L$ and observe that for large $N(L)$ the group G is likely to be trivial. But we care for *infinite* G *and this can be guaranteed with positive probability if* $N(L) \leq \exp L/\beta$ *for large* $\beta \geq \beta_{\mathrm{crit}}(F)$ *for free groups* $F = F_k$, $k \geq 2$, *and non-elementary hyperbolic groups* F *in general.* (I have checked this so far under an additional *lacunarity* assumption allowing, for example, Δ being the disjoint union of finite graphs Δ_i of cardinalities d_i, $i = 1, 2, \dots$, with $d_{i+1} \geq \exp d_i$ and such that the shortest cycle in Δ_i has length $\geq \beta_{cr} \log d_i$ for all i.)

To have an infinite random $G = G(F, \Delta)$ with interesting features, we need a special Δ. We take Δ, such that it contains arbitrarily large λ-*expanders* with a fixed (possibly small) $\lambda > 0$. (Such Δ do exist, in fact random Δ in our category contain such expanders, see [Lub].) Then random groups G a.s. enjoy the following properties.

(A) G are Kazhdan T, i.e. every affine isometric action of such G on the Hilbert space $\mathbb{R}^\infty$ has a fixed point. Furthermore, every isometric action of G on a (possibly infinite dimensional) complete simply connected Riemannian manifold V with $K(V) \leq 0$ has a fixed point (if $\dim V < \infty$, then there is no non-trivial action at all).

(B) For every Lipschitz (for the word metric) map $f : G \to \mathbb{R}^\infty$, there are sequences $g_i,\, g_i' \in G$ with $\mathrm{dist}_G(g_i, g_i') \to \infty$ and $\mathrm{dist}_{\mathbb{R}^\infty}(f(g_i),\, f(g_i')) \leq \mathrm{const} < \infty$ (and the same remains true for the ℓ_p-spaces for $p < \infty$).

One may think the above "pathologies" are due to the fact that G are *not* finitely presented. But one can show that some "quasi-random" groups among our G are recursively presented and so embed into a *finitely presented* group G' which then automatically satisfies (B) and can be chosen with an *aspherical* presentation by a recent (unpublished) result by Ilia Rips and Mark Sapir (where T can be preserved by adding extra relations). Then, one can arrange $G' = \pi_1(V)$ for a closed aspherical manifold V of a given dimension $n \geq 5$ which, besides (B), has more "nasty" features, arresting, for example, all known (as far as I can tell) arguments for proving (strong) Novikov's conjecture for G'. (See [$\mathrm{Gro}_{\mathrm{RW}}$]; but I could not rule out hypersphericity yet.) I feel, random groups altogether may grow up as healthy as random graphs, for example.

There are other possibilities to define random groups, e.g. by following the "symbolic" approach where combinatorial manipulations with finite sets are replaced by parallel constructions in a geometric category. For example, we may give some structure (topology, measure, algebraic geometry) to the generating set B of (future) G with relations being geometric subsets in the Cartesian powers of B. Then, depending on the structure, one may speak of "random" or "generic" groups G (with a possible return to finitely generated groups via a model theoretic reasoning). Looks promising, but I have not arrived at a point of asking questions.

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