An algorithm for the topological type and the determination of analytic components of real algebraic curves *

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Abstract: We present the implementation of an algorithm in AXIOM symbolic computation system, to get topological and geometric informations, as well as the number of analytic components of real algebraic curves in $\widetilde{R}[x, y]$, where $\widetilde{R}$ is a real closure of a real field R. The main idea of the method appears in [CP3R], but this algorithm is not complete and has never been implemented. The present implementation is efficient and it gives the analytic components of some curves not obtained by the latter algorithm. Our implementation is generic.

Keywords: Real algebraic numbers, Real Rational Puiseux Expansions, AXIOM

Contents

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1 Introduction

The aim of this paper is the presentation in AXIOM of a faster algorithm the one proposed by Cucker et al. (see [CP3R]), to analyse branch points at singular points of a real curve, using real rational Puiseux expansions to describe the analytic components of the curve. This algorithm also gives a plane graph homeomorphic to the set :

$$C_p = \{(a, b) \in R^2 | p(a, b) = 0 \text{ \& } p(x, y) \in R[x, y]\}$$

where R is a real field.

Theoretically the problem of parametrization of plane curves is solved, and it is known that the parametrizable curves are exactly the curves of genus 0, see [W], [AB] and [SW]. However, a real algebraic curve can always be locally parametrizable and the algorithm ACRC say how to tracing the irreducible analytic components of the curve.

Moreover, the algorithm ACRC, using the idea of real rational Puiseux expansions and exact algebraic real numbers algorithms, distinguishes the analytic structure of curves that have the same topology; e.g. consider the two curves defined by the following polynomials $p(x, y) := y^2 + x^6 - x^4$ and $q(x, y) := y^4 + x^4 - x^2$, see figure 1.

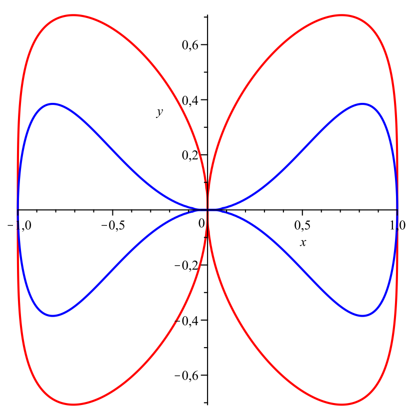


Figure 1: The graphs: $p(x, y)$ blue, $q(x, y)$ red.

These two curves have the same topological graph, shaped roughly like an ∞ (the singular point is the origin of xy-plane); but the first has one analytic component and the second two (its graph looks like two ovals tangent at the origin).

The present algorithm detects two cases where the algorithm in [CP3R] does not:

1. the determination of the order above one real root of the discriminant of $p(x, y)$,
2. the computation of the local topology of the half-branches like the half-branches at the origin of a ramphoid cusp, whose defining polynomial is $2y^4 + (6x - 3)y^3 + (7x^2 - 5x + 1)y^2 + (4x^3 - 2x^2)y + x^4$, see figure 2,

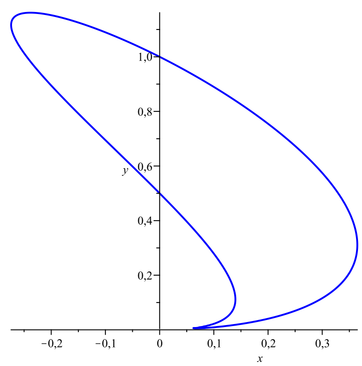


Figure 2: Rampoid cusp.

The argument proposed in [CP3R] that the analytic expansions of $f(x, y)$ are determined in a minimal algebraic extension of R and so it is cheaper as far as computing time is concerned is not valid in practice because it requires a tedious rebuilding of a new tower say, $R_{a_1,\ldots,a_n}$ for further computations. This process it is not efficient, see [L]. We utilize the domain representing the real closure as it appears in [LRR]. We want to emphasize the fact that computations with real algebraic numbers is a costly operation. Our aim is to reduce the number of arithmetic operations in the real closure. For this we will discuss two cases of the computation which differs in the number of arithmetic operations in the real closure: computation over the projections of the singular points, and computation over the projections of the non-singular points. There are some special cases in which a different approach might be preferable, e.g. when C_p is a non-singular curve. Our aim, however, is not a treatement of these special cases, but a general algorithm.

The paper will be divided in three sections. In the first one we briefly review some definitions about the real rational Puiseux expansions; in the second we explain the main algorithm ACRC and give a brief description of a faster algorithm, and in the last section we give some information about the implementation in AXIOM symbolic system, along with examples and time computing bounds.

In this paper we will denote by R a real field, by $\widetilde{R}$ the real closure of R and by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of the rational, real and complex numbers, respectively. For the definitions and the properties of the real fields see [BCR].

2 Definitions – A brief Review

We now recall some definitions concerning real algebraic curves that will be needed later.

Definition 2.1 *Let R be a real field and $p(x, y) \in R[x, y]$ monic in y. A real algebraic curve (r.a.-curve) is the set $C_p := \{(a, b) \in \widetilde{R}^2 \mid p(a, b) = 0\}$. The polynomial $p(x, y)$ is called the defining polynomial for the curve C_p.*

Definition 2.2 *Let C_p be a r.a.-curve. A point $\alpha = (a, b) \in \widetilde{R}^2$ is called a regular or simple point of $p(x, y)$ if not both $p_x(x, y)$ and $p_y(x, y)$ vanish at α. A point $\alpha = (a, b)$ is called a tangency point if $p(a, b) = p_y(a, b) = 0$. A tangency point is called singular if $p(a, b) = p_y(a, b) = p_x(a, b) = 0$. A curve C_p is called regular (or nonsingular) iff every point $(a, b) \in C_p$ is not a singular point; otherwise it is called singular.*

2.1 Real Rational Puiseux Expansions

Let K be a field of characteristic 0 and $\widehat{K}$ any algebraic closure of K. The Puiseux expansions found in the classical way, see [W], are usually not rational, (non invariant under the action of the Galois group $G(\widehat{K((x))}/K((x)))$). D. Duval ([D]) gives another set of Puiseux expansions that are rational. This allows the study of curve singularities over real fields. For example, if $K = \mathbb{Q}$ the roots of the polynomial $p(x, y) := (x^2 + y^2)^3 - 4x^2y^2$ (see [D] p.119) are 6 Puiseux series :

$$\begin{array}{ll} y_1 = \dfrac{1}{2}x^2 + \dots & y_2 = -\dfrac{1}{2}x^2 + \dots \\ y_3 = \sqrt{2}x^{\frac{1}{2}} + \dots & y_4 = -\sqrt{2}x^{\frac{1}{2}} + \dots \\ y_5 = i\sqrt{2}x^{\frac{1}{2}} + \dots & y_6 = -i\sqrt{2}x^{\frac{1}{2}} + \dots \end{array}$$

The rational Puiseux expansions in this case give 4 descriptions of these roots by means of pairs of formal series :

$$
\begin{array}{lll}
x = t & y = \frac{1}{2}t^2 + \ldots & \text{correspond to } y_1 \\
x = t & y = -\frac{1}{2}t^2 + \ldots & \text{correspond to } y_2 \\
x = \frac{1}{2}t^2 & y = t + \ldots & \text{correspond to } y_3 \ \& \ y_4 \\
x = -\frac{1}{2}t^2 & y = t + \ldots & \text{correspond to } y_5 \ \& \ y_6
\end{array}
$$

Let $f(x,y) := \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j}x^j y^i$. Following Newton's algorithm, see [W], the polynomial has n Puiseux expansions $y_1, \ldots, y_n$ at the origin, namely:

$$y_l := \sum_{d=1}^{+\infty} \alpha_{l,d} x^{n_{l,d}/e_l}.$$

For each Puiseux expansion of $f(x,y)$, let

$$x_l(t) := t^{e_l} \quad \text{and} \quad y_l(t) := \sum_{d=1}^{+\infty} \alpha_{l,d} t^{n_{l,d}}.$$

Each pair (x_l, y_l) is a parametrization of C_f $\left(f\left(x_l(x^{1/e_l}), y_l(x^{1/e_l})\right) = 0\right)$. If there does not exist $k > 1$ such that the x_l and y_l are elements of $\widehat{K}[[t^k]]$, then we say that the parametrization (x_l, y_l) is *irreducible*. We say that two parametrizations (x_l, y_l) and (x_m, y_m), are *equivalent*, iff there exists $z \in \widehat{K}[[t]]$ of t-order equal to 1, see [W], such that $x_l(t) = x_m(z(t))$ and $y_l(t) = y_m(z(t))$. A branch (or a place) of C_f is defined as an equivalence class of irreducible parametrizations of C_f, and its center is the center of the parametrizations of the class, see [W], [D].

Definition 2.3 *Let K, $f(x,y)$ be as above and $\{y_l\}_{l=1\ldots n}$ an irreducible parametrization. Let ν be the number of equivalence classes of the set $\{y_l\}_{l=1\ldots n}$. Suppose that $f(0,0) = 0$. A system of rational Puiseux expansions of $f(x,y)$ over K is a set*

$$\{(\widetilde{x}_1, \widetilde{y}_1), \ldots, (\widetilde{x}_\nu, \widetilde{y}_\nu)\}$$

of ν pairwise non-equivalent irreducible parametrizations of C_f, which is invariant under the action of the Galois group $G(\widehat{K}/K)$. Every coefficient of $\widetilde{x}_l$ and $\widetilde{y}_l$ is a number of a finite algebraic extension of K and such that $\forall l$, $\widetilde{x}_l$ is a monomial $c_l t^{e_l}$, $c_l \neq 0$ and $e_l > 0$.

Definition 2.4 *([BCR]) : Let $f(x,y)$ be a irreducible polynomial in $K[x,y]$, $K(C_f) := K(x)[y]/(f(x,y))$ be the quotient field, which is an extension of $K(x)$. Let $(\widetilde{x}_l, \widetilde{y}_l)$ be a parametrization at the origin of $f(x,y)$. We denote by $\widetilde{\mathcal{B}}$ the place in $\widehat{K}(C_f)$ correspoding to $(\widetilde{x}_l, \widetilde{y}_l)$, lying above the place $\mathcal{B}$ of $K(C_f)$. Let $\widetilde{\mathcal{O}}$ be the valuation ring of $\widetilde{\mathcal{B}}$ and $\mathcal{O} = \widetilde{\mathcal{O}} \bigcap K(C_f)$. Let $L := \mathcal{O}/\mathcal{B}$ be the residual field of $\mathcal{B}$. The place $\mathcal{B}$ is real iff its residual field L is real.*

Lemme 2.1 *: Let L be defined as above and L^* the field of coefficients of $(\widetilde{x}_l, \widetilde{y}_l)$. Then, there exists a K-algebra homomorphism from L into L^*.*

Proof (see [D] Lemma section 3) :

There exists a $\widehat{K}$-algebra homomorphism

$$\psi : \widehat{K}[x, y] \to \widehat{K}[[t]],\ g \mapsto g(\widetilde{x}, \widetilde{y})$$

The restriction $\phi : K[x, y] \to L^*[[t]]$ is a K-algebra homomorphism such that $ker(\phi)$ contains no non-zero element of $K[x]$. Since $f(x, y)$ is irreducible we get $ker(\phi) = (f)$.

Thus, we obtain a K-algebra homomorphism

$$\varphi : K(x)[y]/(f(x, y)) := K(C_f) \to L^*((t))$$
$$\text{and } \varphi(\mathcal{O}) \subset L^*[[t]]$$

Let the K-algebra homomorphism

$$\theta : L^*[[t]] \to L^* \text{ such that } \theta(\sum_{i=0}^{+\infty} c_i t^i) = c_0$$

Then $\theta \circ \varphi \mid_{\mathcal{O}}: \mathcal{O} \to L^*$ is a K-algebra homomorphism. Since $ker(\theta \circ \varphi \mid_{\mathcal{O}}) = \mathcal{B}$ it induces a K-algebra homomorphism from $\mathcal{O}/\mathcal{B} = L$ to L^*. ■

The following corollary characterizes a rational Puiseux expansion that is real.

Corollary 2.1 *([D]): Let $(\widetilde{x}_l, \widetilde{y}_l)_l$ be a system of rational Puiseux expansions of $p(x, y) \in \mathbb{R}[x, y]$. Then for each l the branch corresponding to $(\widetilde{x}_l, \widetilde{y}_l)$ is real iff every coefficient of $\widetilde{x}_l$ and $\widetilde{y}_l$ are members of $\mathbb{R}$.*

Proof : Easy by the previous Lemma and definition 4. ■

Technical Remark 1 *Following corollary 1 we can define the real system of rational Puiseux Expansions. It will be seen in section 2.1.1 algorithm 4, that the computation in practice can be made in finite steps in the real closure. The argument proposed in [CP3R] that this computation lies in the smallest one among the coefficient fields of the parametrization, is not efficient, see [L] and [LRR]. So, our computations will lie over the real closure of the ordered field of the coefficients of $p(x, y)$.*

Definition 2.5 *Let $p(x, y)$ be a polynomial in $R[x, y]$ with $p(0, 0) = 0$. A system of rational Puiseux expansions of $p(x, y)$*

$$\{(\widetilde{x}_1, \widetilde{y}_1), \ldots, (\widetilde{x}_\nu, \widetilde{y}_\nu)\}$$

is real iff the coefficients of $\widetilde{x}_l$ and $\widetilde{y}_l$ are elements of the real closure of R.

Example : Let $p(x, y) := y^7 - 3xy^5 + 3x^2y^2 - x^3y + x^4$ be a polynomial in $\mathbb{R}[x, y]$. The system of rational Puiseux expansions that lies above 0 is:

$$\left\{\begin{array}{lll} 1. & x = t & y = \varrho_i t + \dots \\ 2. & x = 9t^5 & y = -3t^2 + \dots \end{array}\right\}$$

where ϱ_i are the complex roots of the polynomial $3x^2 - x + 1$. The curve has one real branch corresponding to the parametrization 2.

The computation of a system of rational Puiseux expansions will be exposed in section 2.1.

2.2 Computing the global analytic components of the curve

(For the definition of the complex and coherent analytic set see [N]. For the concept of semi-algebraic set and semi-algebraic function see [BCR].)

Definition 2.6 *(See [N] p. 104) : Let A be a subset of an open set $\Omega \subset \mathbb{R}^n$, $(\mathbb{R}^n \subset \mathbb{C}^n)$. A is called C-analytic if there exists an open $\Omega' \subset \mathbb{R}^n$, $\Omega' \bigcap \mathbb{R}^n = \Omega$ and a complex analytic set $S \subset \Omega'$ such that $S \bigcap \mathbb{R}^n = A$. Equivalently (see [N] prop. 15 p. 104), iff there are finitely many analytic functions f_i in Ω such that $A = \{x \in \Omega | f_i(x) = 0,$ for all i $\}$*

Consider now, C_p, (definition 1), as a real analytic set. It is well known that C_p is a coherent analytic set and therefore a C-set, (the converse is not true in general) see [BW] or [N]. According to proposition 11 p. 155 [BW], it is possible to decompose C_p as the union of a countable number of C-irreducible, (the union of two C-analytic sets is not different from itself), and C-sets, U_i. This decomposition is unique and incontractible (or irredundant), that is for $i \neq j$ we have $U_i \not\subset U_j$.

Definition 2.7 *(See also [BCR])*

1. *Let U be an open semi-algebraic set of $\mathbb{R}^n$. A semi-algebraic (see [BCR]) function of the class $\mathcal{C}^\infty$ (the class of all infinitelly differentiable with continous semi-algebraic derivatives), $f : U \to \mathbb{R}$ is called a Nash function. If U' is another open semi-algebraic set of $\mathbb{R}^n$, a Nash-diffeomorphism of U into U' is a bijection of U onto U', so that both f and f^{-1} are Nash.*
2. *Let M be a semi-algebraic set of $\mathbb{R}^n$.*
   - (a) *The set M is called a Nash-subvariety of $\mathbb{R}^n$ of dimension d, if $\forall x \in M$ there exists a Nash-diffeomorphism ϕ of a semi-algebraic open neighborhood, $\mathcal{V}$, of x in $\mathbb{R}^n$ into a semi-algebraic open neighborhood, $\mathcal{V}'$, of x in $\mathbb{R}^n$ such that $\phi(0) = x$ and $\phi(\mathbb{R}^d \bigcap \mathcal{V}) = M \bigcap \mathcal{V}'$.*
   - (b) *Two Nash-subvarities M and M' are called Nash-diffeomorhic if there exist a bijection $M \to M'$, so that both f and f^{-1} are Nash.*

3. *Let M be a Nash-subvariety of $\mathbb{R}^n$. A Nash-set in M is a semi-algebraic subset of M in form*

$$\{x \in M | f_1(x) = \ldots = f_p(x) = 0\}$$

where $\{f_1, \ldots, f_p\}$ is a family of Nash-functions of M into $\mathbb{R}$.

Proposition 2.1 *Let C be a real algebraic curve. Then the previous decomposition in C-irreducible and C-sets of C, is finite.*

Proof :

The curve C is a Nash-set and so there exists a decomposition $C = \bigcup_{i=1}^{n} V_i$, where each V_i is an irreducible Nash set, and so a real analytic irreducible set, see proposition 8.6.7 [BCR]. Let U_i be a real analytic set such that $V_i \subset U_i$, $\forall i$, and if there exists E so that $V_i \subset E$ then $U_i \subset E$, (the intersection of any family of analytic sets is still analytic set, see [N]). It can be checked that, the decomposition $C = \bigcup_{i=1}^{n} U_i$ is irredundant and the sets are U_i irreducible. But C is a coherent, and thus a C-analytic set (see [F]). Therefore, following proposition 11 p. 155 [BW], this decomposition is unique. ■

Our algorithm gives a description of these irreducible components of pure dimensional part of C_p. Following [B] we may conjecture that:

> *they analytic irreducible components of the previous decomposition are Nash-sets.*

It is important to note that, there exist curves whose defining polynomials are algebraically irreducible and yet the curves are analytically reducible. For example, take the curve defined by the irreducible polynomial $y^2 - x^2 - x^4$. This curve has a graph roughly shaped like a X, but it has two analytic components.

2.3 Decision procedure for finding the projection of singular points of a curve

In this subsection we establish an one-to-one correspondence between the singular points of a curve (real or complex) and the distinct roots of the discriminant $d(x)$ of the defining polynomial for the curve. This correspondence is described in [S].

Let $p(x, y)$ be a polynomial in $\mathbb{Q}[x, y]$. Let u, v be new coordinates so that $x = u + mv$, $y = u$, and consider $g(u, v) = p(u + mv, v)$; then we can pick an integer m so that the following two conditions are satisfied (see Lemma 2.2, [S]) :

a) $g(u, v)$ is monic.

b) whenever the points (u_0, v_0), (u_0, v_1) satisfy the equations $g = \partial g/\partial u = 0$, then $v_0 = v_1$.

Denote by $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, $g_u = \partial g/\partial u$, $g_v = \partial g/\partial v$.

Consider the polynomial $q(u, v, t) = tg_u - g_v$, and let $a(u, t) = Res_v(g, q)$. We have (see Proposition 2.3 in [S]) :

Let a be as above. Write $a(u, t) = \sum_i a_i(u)t^i$, and consider $A(u) = gcd_i(a_i(u))$. Then

1. C_g is singular over $\mathbb{R}^2$ iff $A(u) = 0$ is solvable in $\mathbb{R}$.
2. C_g is singular over $\mathbb{C}^2$ iff $A(u) = 0$ is solvable in $\mathbb{C}$.

The proof, (see p.39–40 [S]), indicates that the real (complex) singular points are in one-to-one correspondence with the distinct real (complex) roots of $A(u)$. However, in our decision procedure to find the projection of singular points, all we need is a weaker test (which is more efficient since the new polynomial g which satisfies the previous condition b, possesses more cells than the initial polynomial p in the procedure of the Cylindrical Algebraic Decomposition). Thus, a similar procedure can be applied directly to the original polynomial $p(x, y)$ for testing whether C_p is singular over $\mathbb{C}^2$. We caution however, that this test fails to give us a definite answer as to whether C_p is real singular.

Example : Consider the tacnode $p := y^4 - 2y^3 + y^2 - 3x^2y + 2x^4$, see 3. Our defining polynomial for the tacnode satisfies the condition a but not the condition b. The discriminant of p is $d := x^6(2048x^6 - 4608x^4 + 37x^2 + 12)$ which has five real roots. The polynomial $A(u)$ in this case is u^6. So, the projection of the singular points (real or complex) in the x-axis is the origin, $(0, 0)$.

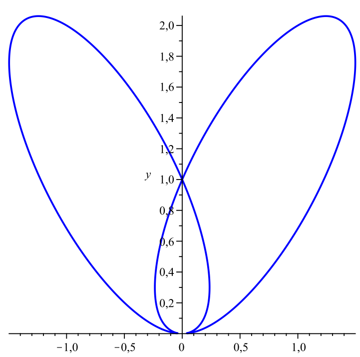


Figure 3: The tacnote $y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0$.

3 The Algorithm ACRC

In this section $f(x,y) \in R[x,y]$ with $f(0,0) = 0$,

$$f(x,y) := \sum_{i=0}^{n} \sum_{j=0}^{m} a_{i,j} x^j y^i$$

and $f(x,y)$ monic a polynomial in y.

3.1 The Computation of a System of Rational Puiseux expansions

In the present section we study the computer algebra construction of the system of rational Puiseux expansions of a plane curve using a variant of Newton's algorithm, see [W], [D]. A parametrization of a branch passing through the origin, of C_f

$$\left(\widetilde{x} = \lambda t^e, \quad \widetilde{y} = \sum_{i=1}^{+\infty} \alpha_i t^i\right)$$

is real iff λ and α_i are real; see corollary 1 and definition 3 section 1.1. The first step in our construction of the algorithm, relates to the computation of a such parametrization corresponding in the real branches. For this computation, we need to compute the coefficients and the exponents of $\widetilde{x}$ and $\widetilde{y}$. First, we define some usefuls technical terms of algorithmic context.

Definition 3.1 *A R-real term ($R - \mathcal{RT}$) is a list*

$$\tau_k := (p_k, q_k, l_k, u_k, v_k, < \varrho >_k, s_k)$$

where p_k, q_k are positive integers, $gcd(p_k, q_k) = 1$ l_k, u_k and $v_k \in \mathbb{Z}$, such that $v_k p_k + u_k q_k = 1$, if R' is an finite extension of R, $< \varrho >_k$ is the representation of the real root ϱ_k of a polynomial $h(x) \in R'[x]$ (this contains the polynomial $h(x)$ and the coding of the real root by interval coding or by Thom's coding, see [LRR]) and $s_k := sign(\varrho_k)$ in R'.

Definition 3.2 *An R-real rational development ($R - \mathcal{RRD}$) is a countable list of $R - \mathcal{RT}_k$*

$$\pi := [\tau_1, \ldots, \tau_k]$$

and there exists a positive integer $k_0 \leq k$ such that $q_i = 1$ and $\varrho_i^{v_i} = 1 \; \forall i \geq k_0$ (for the existence of k_0 see below).

Staring at one edge in the Newton polygon in the first interaction which the equation has real roots, an $R - \mathcal{RT}_k$ is the data corresponding to Newton-like k-interaction to compute the coefficients and the exponents of a real root as a

Puiseux series in x of the polynomial $f(x,y)$. Let ν be the number of equivalence classes of real parametrizations at the origin. The following proposition gives the relation between the $R-\mathcal{RRD}$ and the parametrizations of real branches.

Proposition 3.1 *For each finite $R-\mathcal{RRD}$, $\pi_\nu := [\tau_1, .., \tau_{k_0}]_\nu$ such that $q_{k_0} = 1$ and $\varrho^{v_{k_0}} = 1$, there exists a corresponding parametrization :*

$$P(\pi_\nu) := \Big[\widetilde{x} = \lambda t^e, \quad \widetilde{y} = \sum_{k=1}^{k_0} \alpha_k t^{n_k^{k_0}}\Big]_\nu$$

where for $k \leq k_0$:

$$\begin{cases} e := q_1 q_2 \dots q_{k_0}, \qquad \lambda := \mu_1^{q_0^0} \mu_2^{q_0^1} \dots \mu_{k_0}^{q_0^{k_0-1}} \\ \textit{where } q_i^j := \prod_{k=i+1}^{j} q_k \ \forall i,j : 0 \leq i \leq j \leq k_0, \quad q_i^i := 1, \quad \mu_k := \varrho_k^{-v_k} \\ n_k^{k_0} := p_1 q_1^{k_0} + p_2 q_2^{k_0} + \dots + p_k q_k^{k_0}, \quad \alpha_k := \varrho_k^{u_k} \beta_k \quad \textit{where} \\ \beta_k \ := \ (\mu_2 \mu_3^{q_2} \dots \mu_{k_0}^{q_2 \cdots q_{k_0-1}})^{p_1} (\mu_3 \mu_4^{q_3} \dots \mu_{k_0}^{q_3 \cdots q_{k_0-1}})^{p_2} \dots \\ \qquad\qquad (\mu_{k+1} \mu_{k+2}^{q_{k+2}} \dots \mu_{k_0}^{q_{k+1} \cdots q_{k_0-1}})^{p_k} \end{cases}$$

(for any k, see [D].)

Proof :

Consider new symbols $x_0, x_1, \dots, x_{k_0}$ and $y_0, y_1, \dots, y_{k_0}$, and the relations

$$(\mathcal{S}_k) \qquad \begin{cases} x_{k-1} &= \varrho_k^{-v_k} x_k^{q_k} \\ y_{k-1} &= (\varrho_k^{u_k} + y_k) x_k^{p_k}, \quad 1 \leq k \leq k_0 \end{cases}$$

The proposition follows by the elimination of $x_1, \dots, x_{k_0}$ and $y_1, \dots, y_{k_0}$. ■

– DESCRIPTION

In the description of the algorithm we use the *Newton polygon* of a polynomial $f(x,y)$. This is defined as the lower part of the convex hull of the set of points (i,j) such that $a_{i,j} \neq 0$ along with the origine, see also [W].

Algorithm 1 COEFFICIENTS

input : *The polynomial* $f(x,y)$.

output : *A list of integers* $(i, o(\sum_{j=j_0}^{m} a_{i,j} x^j))$, *where* $o(\sum_{j=j_0}^{m} a_{i,j} x^j) := j_0$ *if* $a_{i,j_0} \neq 0$.

Algorithm 2 BEZOUT

input : *Two integers* p *and* q *such that* $gcd(p, q) = 1$.

output : *Two integers* u *and* v *such that* $up + vq = 1$.

This is the well known extended Euclidean algorithm. If $p = 1$ then $(u, v) := (1, 0)$.

Algorithm 3 CONVEXHULL

input : *A list of points* $lp := [(i_d, j_d)_d]$, $(i_d, j_d) \in \mathbb{Z}^2$.

output : *The list of list of points of edges of segments in the Newton polygon.*

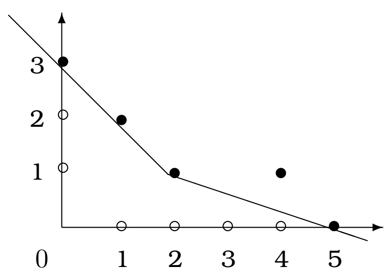
Algorithm 4 RATEXPAN

input : *A polynomial* $f'(x, y) \in R'[x, y]$, *where* R' *is a finite algebraic extension of* R *such that* $f'(0, 0) = 0$.

output : *A finite list of* $R - \mathcal{RT}$. *This list contains all the "real" informations by the Newton polygon analysis applied in the* $f'(x, y)$.

Example : Let $f(x, y) := y^5 + 2xy^4 - xy^2 - 2x^2y + x^4 - x^3$. As the first step, the procedure RATEXPAN finds the COEFFICIENTS of $f(x, y)$, (algorithm 1), and the corresponding CONVEXHULL (algorithm 3):

$$lp := \text{COEFFICIENTS}(f) := [(0, 3), (1, 2), (2, 1), (4, 1), (5, 0)]$$
$$ch := \text{CONVEXHULL}(lp) := [[(0, 3), (1, 2), (2, 1)], [(2, 1), (5, 0)]]$$



Let $\Delta_1 := [(0, 3), (1, 2), (2, 1)]$. Then $3 + 0\gamma = 2 + 1\gamma \Rightarrow \gamma = 1/1$. So, $p = q = 1$, $u = 1$, $v = 0$, $l = 3$ and $i_0 := min(0, 1, 2) = 0$. The characteristic polynomial is $h(z)_{\Delta_1} := -z^2 - 2z - 1 = -(z + 1)^2$ and it has only one real root $\varrho_1 := -1$. Then

$$\tau_1 := (1, 1, 3, 1, 0, < \varrho_1 >, -1).$$

Let $\Delta_2 := [(2, 1), (5, 0)]$, $1 + 2\gamma = 0 + 5\gamma \Rightarrow \gamma = 1/3$. Then, $p = 1$, $q = 3$, $u = 0$, $v = 1$, $l = 5$ and $i_0 := min(2, 5) = 2$. The corresponding characteristic polynomial is $h(z) := z - 1$ and it has one real root $\varrho_2 := 1$ and the corresponding $R - \mathcal{RT}$ is

$$\tau_1' := (1, 3, 5, 0, 1, < \varrho_2 >, 1).$$

Finally, the algorithm gives the list of $R - \mathcal{RT}$: $[\tau_1, \tau_1']$.

Algorithm 5 NEWPOLYNOMIAL

input : *A polynomial $f'(x,y) \in R'[x,y]$, where R' is a finite algebraic extension of R and a $R - \mathcal{RT}$ $(p,q,l,u,v,< \varrho >,s)$.*

output : *The polynomial*
$$g(x,y) := x^{-l} f'(\varrho^{-v} x^q, (\varrho^u x^p + x^p y)) \in R'(\varrho^{-v}, \varrho^u)[x,y]$$

Algorithm 6 RealRationalExpansions (abbrev. RREXP)

Let $f(x,y)$ be a polynomial in $R[x,y]$. We denote by $R - \mathcal{RRD}_f$ the list of the first k terms of the type $R - \mathcal{RT}$ of the $R - \mathcal{RRD}$, $\pi := [\tau_1, \ldots, \tau_k \ldots]$, of the polynomial $f(x,y)$, such that the real root ϱ_k in the last τ_k is a simple zero. Then, the branch corresponding to $R - \mathcal{RRD}_f$ is real, according to corollary 1 section 2.2, because in each $i-$interaction, $i > k$, the segment of negative slope has horizontal length at most m, where m is the multiplicity of the real root in the $i-1$ interaction (see [W]), and so in our case, ($m = 1$), the degree of the characteristic polynomial is equal to 1 (the finitness of algorithm). It is also easy to verify that for each $i > k$, $q_i = 1$.

input : *The polynomial $f(x,y)$.*
output : *A finite list of $R - \mathcal{RRD}_f$. The length of this list is equal to ν, where ν is the number of equivalence classes of real parametrizations or equivalently the number of real branches at the origin of the polynomial $f(x,y)$.*

Examples : In the following examples, the coding method used for the real algebraic numbers, is the interval coding method, see [LRR].

1) Let $f(x,y) := y^5 + 2xy^4 - xy^2 - 2x^2y + x^4 - x^3$. The algorithm RealRationalExpansions runs the following list of real branches passing through the origin :

$$TotRBr_{(0,0)} := [\pi_1, \pi_2]$$

where :
$\pi_1 := [(1,1,3,1,0,< -x^2 - 2x - 1, -1 >, -1), (1,2,0,1,< -x+1, -1 >, 1)]$
$\pi_2 := [(1,3,5,0,1,< x-1, 1 >, 1)]$

The curve has two real branches passing through the origin defined by the $R - \mathcal{RRD}$, π_1 and π_2. The first real branch is coded by two $R - \mathcal{RT}$ because the first has not a simple real root.

2) The polynomial defining the ramphoid cusp : $f(x,y) := 2y^4 + (6x-3)y^3 + (7x^2 - 5x + 1)y^2 + (4x^3 - 2x^2)y + x^4$.

$$TotRBr_{(0,0)} := [\pi_1]$$

where :
$\pi_1 := [(2,1,4,1,0,< x^2 - 2x + 1, 1 >, 1), (1,2,2,0,1,< x-1, 1 >, 1)]$

So, the curve has one real branch π_1 at the origin coded by two $R - \mathcal{RT}s$.

As an immediate consequence of this algorithm we present another algorithm, (we call it RealRationalPuiseuxSeriesQ), which computes the n-first terms of all the real Puiseux series.

Algorithm 7 RealRationalPuiseuxSeriesQ (abbrev. RRPSQ)

input : *A polynomial* $f(x, y) \in R[x, y]$, *and an integer* n.

output : *A finite list of* $R - \mathcal{RRD}_f$. *The length of this list is equal to* ν, *where* ν *is the number of equivalence classes of real irreducibles parametrizations or equivalently the number of real branches at the origin of the polynomial* $f(x, y)$. *Every* $R - \mathcal{RRD}_f$ *has exactly* n $R - \mathcal{RT}s$.

3.2 The Main Algorithm

As we have seen, (cf. 1.2), our algorithm gives a description of the real analytic and irreducible components of $f(x, y)$. The strategy involves the use of real rational Puiseux expansions. The algorithm is structured in three phases :

PHASE 1 : TRANSLATION

1. We compute the discriminant locus and the branch points (simple, tangency simple and singular points) $(\xi_i, \zeta_{i,l_i})_i$ of the curve C_f.
2. We find the projections of singular points of the curve. Let $(\xi_k, \zeta_{k,l_k})_k$ be these points.
3. We translate the point (ξ_k, ζ_{k,l_k}) to the origin. Let $f'(x, y) := f(x+\xi_k, y+\zeta_{k,l_k})$ be the new polynomial.
4. We compute the RREXP$(f'(x, y))$ for each branch point.

PHASE 2 : COMPUTING THE LOCAL ANALYTIC COMPONENTS – ADJACENCY RELATION

1. Compute the simple points and tangency simple points, over the real roots $\xi_{i'}$ for $i' \neq k$ for all k that satisfy the second condition in PHASE 1.
2. (a) Compute the polynomial $g(x, y) := \dfrac{\partial^2 f}{\partial y^2}$. If the sign of this polynomial at the tangency simple point is zero go to (c), else go to (b)
   (b) Compute the polynomial $h(x, y) := \dfrac{\partial f}{\partial x} f(x, y)$, and the sign of the $h(x, y)$ at the tangency simple points.
   (c) Compute the local topology at the simple points.
   (d) Compute the local topology by the sign of the $h(x, y)$, at tangency simple points.

3. Compute the local topology of the real points over the $\xi_k' s$.

4. Compute the adjacency relations of half-branches of every branch point.

PHASE 3 : COMPUTING THE GLOBAL ANALYTIC COMPONENTS

1. Describe the global analytic components of the given curve.

3.2.1 The Algorithm of the PHASE 1

Algorithm 8 : SING

input : *The monic polynomial $f(x, y)$ and its discriminant $d(x)$ with respect to y.*

output : *The list of the real roots of $d(x)$ into two classes : the class of the projections of the complex, in general, singular points and the class of the projections of the real simple and simple tangency points.*

– DESCRIPTION (see also **1.3**)

- Let $q(x, y, w) := wf_x - f_y$, $f(x, y, w) := f(x, y)$, and $a(x, w) :=$ $Res_x(f(x, y, w), q(x, y, w))$. If $a(x, w) := \sum_i a_i(x)w^i$, let $h(x) := gcd_i(a_i)$.
- We find the real solutions of the system $d(x) = h(x) = 0$. We denote by $\{\xi_k\}_{1\leq k\leq s}$ these commun solutions, where s is the number of real solutions of the $d(x)$.

Technical Remark 2 *The algorithm that find the real solutions of $d(x) = h(x) = 0$, varies with the method of the real root coding; so, in Thom's coding method we utilize a special case of the algorithm for coding the real roots of a polynomial (see [L]), and in the interval coding method we evaluate $h(x)$ over the real roots of $d(x)$.*

Algorithm 9 TRREXP

input : *The polynomial $f(x, y)$*

output : *A finite list of list of $R - \mathcal{RRD}_f$, see algorithm 1. The size of this list is equal to k. Each list of $R - \mathcal{RRD}_f$ contains the real rational Puiseux expansions in the real points (ξ_k, ζ_{k,l_k}), where ζ_{k,l_k} are the real solutions of $f(\xi_k, y) = 0$.*

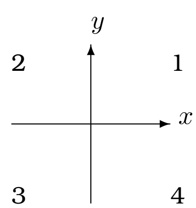
– DESCRIPTION

- Let $d(x)$ be the discriminant of $f(x, y)$ with respect to y. After a square free reduction we assume that $d(x)$ has no repeated roots. We code the real roots $\{\xi_1, \ldots, \xi_s\}$ of $d(x)$.
- For each t, $1 \leq t \leq s$, let $\{\zeta_{t,1}, \ldots, \zeta_{t,l_t}\}$ be the real roots of $f(\xi_t, y) = 0$.
- We characterize the real roots of $d(x)$ into classes, the projections of real or complex singular points, and the projections of real simple and simple tancency points, following the previous algorithm SING. Let ξ_k be the projections of singular points.
- We translate the point (ξ_k, ζ_{k,l_k}) to the origin. Let $f'(x, y) := f(x+\xi_k, y+\zeta_{k,l_k})$ be the new polynomial.
- We compute the RREXP$(f'(x, y))$ for each branch point.

3.2.2 The Algorithm of the PHASE 2

The local topology over the points $\xi_{i'}$, $i' \neq k$

In the following the quadrant $(x > 0, y > 0)$ will be labelled by 1, the $(x < 0, y > 0)$ by 2, the $(x < 0, y < 0)$ by 3 and the $(x > 0, y < 0)$ by 4.



Let (a, b) be a non-singular point of the curve. If this point is a simple point we say that the corresponding real branch lies in the quadrants $2 \to 4$; this position has the same final effect as the position $1 \to 3$; our choice is accidental. Now suppose that (a, b) is a simple tangency point; if the sign of the polynomial $g(x, y)$, see PHASE 2 2.(a), in this point is zero, then the polynomial has an inflexion point, and we can say that the real branch lies in the quadrants $1 \to 3$. Else, the sign of the polynomial $h(x, y)$, (see PHASE 2, 2(a)), is non zero, it is positive iff the curve has two real branches on the left, i.e. local position $2 \to 3$, and it is negative iff the curve has two real branches on the right, i.e. local position $1 \to 4$.

Example : Consider the curve $p(x, y) := y^6 + (x^4 + x^3 - 1)y^3 + 3x^3 - x^2 + 2x$, see figure 4. The discriminant of $p(x, y)$ has three real roots. Over the first root, we can see that the polynomial has an inflexion point, (the polynomial $f(x, y)$ is zero), over the second root the simple tangency point make the polynomial $h(x, y)$ positive, and over the last root the corresponding simple tangency point makes the polynomial $p(x, y)$ negative.

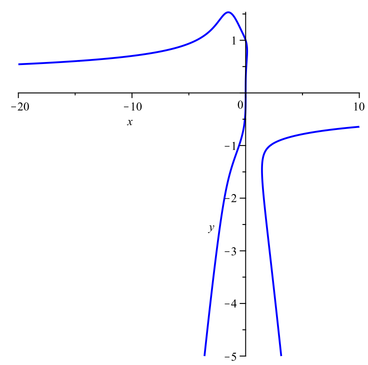


Figure 4: $p(x,y) := y^6 + (x^4 + x^3 - 1)y^3 + 3x^3 - x^2 + 2x$

The local topology over the points ξ_k

Let $\pi := [\tau_1, .., \tau_{k_0}]$ be an $R - \mathcal{RRD}$, that determines a real branch. Let $\tau_1 := (p_1, q_1, l_1, u_1, v_1, < \varrho_1 >, s_1)$ be the first $R - \mathcal{RT}$ of π.

If the corresponding real branch is determined by only the first $R - \mathcal{RT}$ τ_1, then the parities of p_1 and q_1 and the sign s_1 are sufficient to determine in which quadrants the real branch Br_r lies, according to the relation $(\mathcal{S}_1)$ section 2.1. These conclusions are summarized in the following table to facilitate qualitative inferences from the sign of ϱ and the parities of p and q (see also [CP3R] or [L]):

p	**q**	sign(ϱ) = s	**quadrants**
odd	odd	positive	3→ 1
		negative	4→ 2
odd	even	positive	4→ 1
		negative	3→ 2
even	odd	positive	2→ 1
		negative	4→ 3

table 1

Proposition 3.2 *If a real branch is determined by the first $R-\mathcal{RT}$, then its local position at the origin, is given by the table 1.*

Proof :

(*Notation:* The notation $(\widetilde{x}, \widetilde{y}) \in 1$, 2, 3 or 4, signifies that the real branch lies in the quadrant 1, 2, 3 or 4 respectively.)

We restrict our attention to the case where $sign(\varrho) = -1$; the case $sign(\varrho) = +1$ is the same. If $sign(\varrho) = -1$, we have the following cases : Let ϵ be an positive infitesimal.

• *case 1* $- p, q :$ odd and $s = sign(\varrho) = -1$. Then,

$$\begin{array}{cc} sign(-\epsilon)^q = -1 & sign(-\epsilon)^p = -1 \\ sign(\epsilon)^q = +1 & sign(\epsilon)^p = +1 \end{array}$$

But, $sign(\varrho^{-v}) = sign(\varrho^{-vp})$ and $sign(\varrho^u) = sign(\varrho^{uq})$. So,

$$\begin{aligned} sign(\varrho^{-v})sign(\varrho^u) &= sign(\varrho^{-vp})sign(\varrho^{uq}) \\ &= sign(\varrho^{1-2vp}) \\ &= sign(\varrho) = -1 \end{aligned}$$

If $sign(\varrho^{-v}) = -1$ and $sign(\varrho^u) = +1$ by the $(\mathcal{S}_1)$ section 2.1,

$$\text{For } t = \epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = -1 \\ sign(\widetilde{y}) = +1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 2$$

$$\text{For } t = -\epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = +1 \\ sign(\widetilde{y}) = -1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 4$$

If $sign(\varrho^{-v}) = +1$ and $sign(\varrho^u) = -1$:

$$\text{For } t = \epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = +1 \\ sign(\widetilde{y}) = -1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 4$$

$$\text{For } t = -\epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = -1 \\ sign(\widetilde{y}) = +1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 2$$

• *case 2* $- p :$ odd $q :$ even, $s = sign(\varrho) = -1$. Then,
$sign(-\epsilon)^q = +1$, $sign(-\epsilon)^p = -1$, $sign(\epsilon)^q = +1$ and $sign(\epsilon)^p = +1$. But now we have, $sign(\varrho^{-v})sign(\varrho^u) = +1$.
If $sign(\varrho^{-v}) = +1$ then v is even and so $uq + vp$ is also even, a contradiction. So, $sign(\varrho^{-v}) = sign(\varrho^u) = -1$.

$$\text{For } t = \epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = -1 \\ sign(\widetilde{y}) = -1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 3$$

$$\text{For } t = -\epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = -1 \\ sign(\widetilde{y}) = +1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 2$$

• *case 3* $- p :$ even $q :$ odd, $s = sign(\varrho) = -1$. As in the case 2,

$$\begin{array}{c} sign(-\epsilon)^q = -1, \quad sign(-\epsilon)^p = +1 \\ sign(\epsilon)^q = +1, \quad sign(\epsilon)^p = +1 \\ sign(\varrho^{-v}) = sign(\varrho^u) = -1 \end{array}$$

$$\text{For } t = \epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = -1 \\ sign(\widetilde{y}) = -1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 3$$

$$\text{For } t = -\epsilon \Rightarrow \left\{ \begin{array}{c} sign(\widetilde{x}) = +1 \\ sign(\widetilde{y}) = -1 \end{array} \right\} \Rightarrow (\widetilde{x}, \widetilde{y}) \in 4$$

■

Remark 1 *The above description does not depend on the choices of the solutions u and v of $up + vq = 1$.*

If the real branch is determined by a series of k_0 $R - \mathcal{RT}$, $k_0 > 1$, then the situation is more complicated. Essentially, we determine the points like a “ramphoid cusp”, if they exist, see figure 2. The procedure is the following :

step 1: First, we evaluate the RREXP($f(x,y)$). Let $\pi := [\tau_1, .., \tau_{k_0}]$ be a real branch through the origin. By the parities of p_1 and q_1 and the sign of ϱ_1 we determine the local topology of π as previously.

step 2: Consider the previous real branch π. Let $\widetilde{x} := \varrho_1^{-v_1} \varrho_2^{-v_2 q_0^1} \ldots$-$\varrho_{k_0}^{-v_{k_0} q_{k_0}^{k_0-1}} t^{q_1 q_2 \ldots q_{k_0}}$, where $q_0^j := \prod_{k=1}^{j} q_k$, $\forall j : 0 < j \leq k_0$.

{ (a) Let the real branch lie in one of the following quadrants $3 \to 1$, $2 \to 1$, $2 \to 4$ or $3 \to 4$. If for $t = \epsilon$ and $t = -\epsilon$ the sign of $t^{q_1 q_2 \ldots q_{k_0}}$ is the same (this is the case, if $q_1 q_2 \ldots q_{k_0}$ is even), then the real branch at the origin has a shape like the ramphoid cusp.

{ (b) If not, the local position of the real branch is given by step 1.

step 3: In this step, we examine the situation of the step 2 (a). Let ϵ be a positive infinitesimal. In this case we easily obtain the sign of $\widetilde{x}$ by the signs of $\varrho_1, \ldots, \varrho_{k_0}$ and the parities of $q_1, \ldots, q_0^{k_0-1}$ and $v_1, \ldots, v_{k_0}$.

step 4: In the other cases, i.e. if $4 \to 1$ or $3 \to 2$, we find the sign of the first term, for $t = \pm\epsilon$, of the series $\widetilde{y}$ of the prop. 2, which is:

position obtain	sign($\widetilde{x}$)	sign($\widetilde{y}$)	**local position**
by step 1			**of the half-brs**
3→ 1	positive		1→ 1
	negative		3→ 3
2→ 1	positive		1→ 1
	negative		2→ 2
4→ 2	positive		4→ 4
	negative		2→ 2
4→ 3	positive		4→ 4
	negative		3→ 3
4→ 1		positive	1→ 1
		negative	4→ 4
3→ 2		positive	2→ 2
		negative	3→ 3

table 2

$$\left(\mu_2\mu_3^{q_2}\dots\mu_{k_0}^{q_2\dots q_{k_0-1}}\right)^{p_1} t^{p_1 q_2\dots q_{k_0}}, \quad \mu_j := \varrho_j^{-v_j}$$

Examples : (1) Let $f := 2y^4+(6x-3)y^3+(7x^2-5x+1)y^2+(4x^3-2x^2)y+x^4$, see figure 2. The curve has one real branch at the origin coded by π_1, see Ex. 2 of the algorithm 6. By the first $R-\mathcal{RT}$, $(2,1,4,1,0,< x^2-2x+1,1>,1)$, and the table 1, we conclude that this real branch lies in the quadrants $2 \to 1$. In this case we have, $p_1 = 2$, $q_1 = 1$, $u_1 = 0$, $v_1 = 1$, $\varrho_1 = 1$ and $p_2 = 1$, $q_2 = 2$, $u_2 = 1$, $v_2 = 0$, $\varrho_2 = 1$. Applying the test in the previous step 2 (a), we see that $q_1q_2 = 2$, so at the origin the curve has a ramphoid point. By prop. 2, $\widetilde{x} = t^2$ or $\widetilde{x}$ is positive for $t = \pm\epsilon$. Finally, by the table 2 the real branch lies in the quadrant 1.

(2) Let $f(x,y) := 2y^5 - xy^3 + 2x^2y^2 - x^3y + 2x^5$, see figure 5.

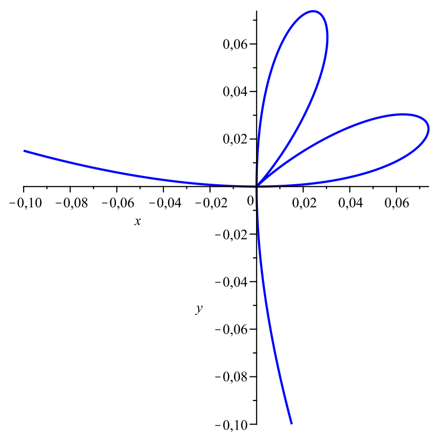


Figure 5: The curve $2y^5 - xy^3 + 2x^2y^2 - x^3y + 2x^5 = 0$.

The alg. 6 gives:
RREXP($f(x,y)$):=$[\pi_1, \pi_2, \pi_3]$:=

$$\Bigg[\Big((2,1,5,1,0,< -x+2,2>,1)\Big),$$
$$\Big((1,1,4,1,0,< -x^2+2x-1,1>,1),(1,2,2,0,1,< -x+4,4>,1)\Big),$$
$$\Big((1,2,5,0,1,< 2x-1,\frac{1}{2}>,1)\Big)\Bigg]$$

The real branches corresponding to the π_1 and π_3 lie in the quadrants $2 \to 1$ and $4 \to 1$ respectively. The real branch coded by π_2 looks like a ramphoid point at the origin, because $q_1 q_2 = 2$, see figure 6. More precisely, this real branch lies in the quadrant $1 \to 1$, because $\widetilde{x} = \varrho_1^{-v_1} \varrho_2^{-v_1 q_1} t^{q_1 q_2} = t^2 > 0$ for $t = \pm\epsilon$.

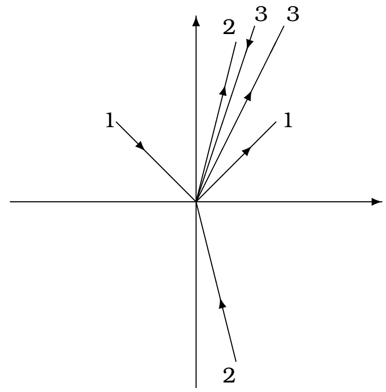


Figure 6

For instance, by the tables 1 and 2 we have the contiguous real half-branches in the real branch points. The adjacency relation between two successive real roots ξ_i and ξ_{i+1} of the discriminant, follows the Cylindrical Algebraic Decomposition analysis and so we must compare all the half-branches which are on the same side of a real root of the discriminant.

Let $\pi_1, \ldots, \pi_r$ denote the half-branches on the same side of a real root ξ_i, left or right. The procedure of the order is the following:
(*Notation:* We note by $\pi_{j,t}$ and by $\pi_{j',t'}$, the real half-branches of π_j and $\pi_{j'}$ which lie in the quadrants t and t' respectively.)

1. if the branch points $\zeta_{i,k}$ and $\zeta_{i,k'}$ corresponding to the half-branches $\pi_{j,t}$ and $\pi_{j',t'}$, $1 \leq j, j' \leq r$, are in order $\zeta_{i,k} < \zeta_{i,k'}$, then $\pi_{j,t} < \pi_{j',t'}$,

2. if $\pi_{j,t}$ belongs in the quadrants 1 (or 2) and $\pi_{j',t'}$ in the quadrants 4 (or 4), then $\pi_{j',t'} < \pi_{j,t}$,

3. in the other case, let $\pi_{j,t} :=< \tau_{1,j}, \ldots, \tau_{k,j} >$ and $\pi_{j',t} :=< \tau_{1,j'}, \ldots, \tau_{k',j'} >$ be two real half-branches in the same quadrant t. Let l be the smallest index such that $\tau_{l,j} \neq \tau_{l,j'}$ and $\tau_{i,j} = \tau_{i,j'}$ $\forall i < l$.

   (a) If $1 < l$,

      i. If $n_l^k < n_l^{k'} \Rightarrow \pi_{j',t} < \pi_{j,t}$

      ii. If $n_l^k = n_l^{k'}$ then $\pi_{j,t} < \pi_{j',t}$ iff

      $$sign\Big(\widetilde{y}(\sigma) - \widetilde{y}'(\sigma)\Big) = -1$$

      where, $\widetilde{y}$ and $\widetilde{y}'$ are the series of the prop. 2, σ is equal to $-\epsilon$, a negative infinitesimal, if $t = 2$ or 3, or $+\epsilon$, a positive infinitesimal, if $t = 1$ or 4. Note that the sign of the difference $\Big(\widetilde{y}(\sigma) - \widetilde{y}'(\sigma)\Big)$ is equal to the sign of $\alpha_l t^{n_l^k} - \alpha_l' t^{n_l^{k'}} = (\alpha_l - \alpha_l') t^{n_l^k}$, for $t = \sigma$.

(b) else, i.e. $l = 1$, see the following proposition 4.

Remark 2 *The proof of conditions of 3.(a).i and ii, is easy.*

Proposition 3.3 *Let*
$y(x) := {\xi_{1,j}}^{1/q_{1,j}} x^{p_{1,j}/q_{l,j}} + \ldots$ *and* $y'(x) := {\xi'_{1,j'}}^{1/q_{1,j'}} x^{p_{1,j'}/q_{l,j'}} + \ldots$ *are the first terms of the compact forms of* $\pi_{j,t}$ *and* $\pi_{j',t}$ *respectively, then*

1. *if* $p_{1,j}/q_{1,j} = p_{1,j'}/q_{1,j'}$, *and* σ *as previously,* $\pi_{j',t} < \pi_{j,t}$ *iff* $sign\Big(y(\sigma) - y'(\sigma)\Big) = +1$, *where* $\sigma = +\epsilon$ *if* $t = 1$ *or 4, or* $\sigma = -\epsilon$ *if* $t = 2$ *or 3, which is equivalent to:*
$$\text{if} \left\{\begin{array}{lll} \xi'_{1,j'} < \xi_{1,j} & \text{and} & \\ i = i' = 4, & j = j' = 1, & t = t' = 4 \\ \text{or} & & \\ i = i' = 3, & j = j' = 2, & t = t' = 2 \\ \text{or} & & \\ i = i' = 3, & j = j' = 1, & t = t' = 3 \\ \text{or} & & \\ i = i' = 4, & j = j' = 2, & t = t' = 2 \end{array}\right\} \text{then, } \pi_{j,t} < \pi_{j',t}.$$
else, if $\xi_{1,j} < \xi'_{1,j'}$ *we have that* $\pi_{j,t} < \pi_{j',t}$.

2. *If* $p_{1,j}/q_{1,j} \neq p_{1,j'}/q_{1,j'}$, *then,*
$$\pi_{j,t} < \pi_{j',t} = \left\{\begin{array}{lll} \textit{true} & \textit{iff} & p_{1,j'}/q_{1,j'} < p_{1,j}/q_{1,j} \\ & & \\ \textit{false} & \textit{else} & \end{array}\right.$$

Proof:

1. We prove only the first.
$sign(y(x) - y'(x)) = sign(-\xi_{1,j}^{1/q_{1,j}} + {\xi'_{1,j'}}^{1/q_{1,j'}}) = -1$ iff $\xi'_{1,j'} > \xi_{1,j}$.
2. Easy. ∎

Example : Consider the curve in the previous two examples. We have nothing to do in the first curve. In the latter, let $\pi_{1,2}$, $\pi_{1,1}$ are the half-branches of π_1 in the quadrants 2 and 1 respectively, $\pi_{3,1}$, $\pi_{3,4}$ are the half-branches of π_3 in the quadrants 1 and 4 respectively, and $\pi_{2,1}^2$ the two half-branches of π_2 in the quadrant 1. Then, the orders in the right side of the origin, are as follows (see also (figure 6)):
(1) $\pi_{3,4} < \pi_{1,1}, \pi_{2,1}^2, \pi_{3,1}$ (following the condition 2).
(2) The compact forms of π_1, π_2 and π_3 are $y_1(x) = 2x^2 + \ldots$, $y_2(x) = x + \ldots$ and $y_3(x) = (\frac{1}{2})^{1/2} x^{1/2} + \ldots$ respectively. By prop 4 (2), $\pi_{1,1} < \pi_{2,1}^2$ (since $1 < 2$) and $\pi_{2,1}^2 < \pi_{3,1}$ (since $1/2 < 1$).
So, we have $\pi_{3,4} < \pi_{1,1} < \pi_{2,1}^2 < \pi_{3,1}$.

The Adjacency Relation

Now, the adjacency relation between two successives real roots of the discriminant ξ_i and ξ_{i+1} is a straightforward consequence of the order of half-branches just right of ξ_i and just left of ξ_{i+1}. We code the real branch points and the real half-branches as follows :

> **1)** a real point by A_{k,l_k} which corresponds to the real root ζ_{k,l_k}, see algorithm of the phase 1.
>
> **2)** we code a real half-branch, left or right of real roots ξ_k, $1 \le k \le s$, see algorithm of the phase 1, by $Br_{(i_1,i_2,i_3,i_4,i_5)}$, where i_1 and i_2 are the quadrants which the real branch lies, i_3 is the enumeration of the real branches in the same quadrants i_1 and i_2, i_4 is the quadrant in which the real half-branch lies and i_5 is its order determined by the previous phase of half-branch left or right of a real root of the discriminant.

The adjacency relation between two real half-branches $Br_{(t,r,l,m,n)}$ and $Br'_{(t',r',l',m',n')}$, is determined by the following :

- **i)** if they are on the left and on the right of the branch point A_{i,j_i} then they belong to the same real branch *iff* $t = t'$, $r = r'$ and $l = l'$.
- **ii)** if they are on the right (or on the left) of the point A_{i,j_i} and on the left of the point $A_{i+1,j_{i+1}}$ (or on the right $A_{i-1,j_{i-1}}$) respectively, then they are adjacent *iff* $n = n'$.

Algorithm 10 ORDERBr

input : *The output of the algorithm*
TRREXP*($f(x,y)$), (see algorithm 9.)*

output : *A lexicographically ordered set :*

$$(\Sigma): \quad \Big\{\ldots, \Big((\ldots Br_{(t,r,l,m,n)},\ldots)_{left}, A_{i,j_i},\\ (\ldots, Br'_{(t',r',l',m',n')},\ldots)_{right}\Big)_{i,j_i},\ldots\Big\}$$

where the lists of left and right half-branches are ordered as previously.

3.2.3 The Algorithm of the PHASE 3

We can finally draw the one-dimensional analytic component of the real curve. The algorithm called TRANSCLOSURE:

Algorithm 11 TRANSCLOSURE

input : *The set* (Σ)

output : *A list of one-dimensional analytic component of the real curve.*

– DESCRIPTION

Let $\mathcal{AC}$ be an one-dimensional analytic component of the real curve.

1) If an element of (Σ) has only a point A_{i,j_i} then this point is a trivial analytic component of the curve.

2) If the set of left half-branch over the ξ_0 is empty go to step **3**. Let Br be the first half-branch over the interval $(-\infty, \xi_0)$ (or $(\xi_s, +\infty)$) centered in the point A_{0,j_0} (or A_{s,j_s}), then Br and $A_{0,j_0} \in \mathcal{AC}$ (or A_{s,j_s}). We go to other side of ξ_0 (or ξ_s) and we repeat step **3.1**.

3) Let $(\ldots, Br, \ldots)$ be the first non-empty list of half-branches left or right over the real root ξ_i. Let Br be the first half-branch in the previous list and A_{i,j_i} the center of Br. Then, $A_{i,j_i}, Br \in \mathcal{AC}$.

3.1) if Br' is the other half-branch adjacencies in Br following the relation **i**, subsection 4.2.2, $Br' \in \mathcal{AC}$. Now, we go to the list labeled by $A_{i+1,j_{i+1}}$ (or $A_{i-1,j_{i-1}}$) and let Br'' be the half-branch which is adjacent to Br' following the relation **ii**, subsection 4.2.2. We continue the above procedure until we find a half-branch which coincides with Br or goes to infinity following the previous case **2**. $A_{i+1,j_{i+1}}$ (or $A_{i-1,j_{i-1}}$), $Br'' \in \mathcal{AC}$. We drop from the set (Σ) the : $Br, Br', Br'', A_{i,j_i}, A_{i+1,j_{i+1}}$ or $(A_{i-1,j_{i-1}})$.

4) We repeat the above procedure until the set (Σ) is empty.

We need to show that the above algorithm gives the one-dimensional real analytic components of the real curve. The component $\mathcal{AC}$ is a real analytic set of C_f (because is locally so at every point), it is also one-dimensional and so C-analytic, coherent (see [F]), and C-irreducible set (because all the contiguous real half-branches belong to same irreducible real analytic local component).

Example : Here, we give a complete example of our algorithm. Let the Ruiz-Castizo's quatric $p := (y^2 - 2x^2 - 3x)^2 - 4x^2(1-x)(2-x)$, see figure 6 and 7.

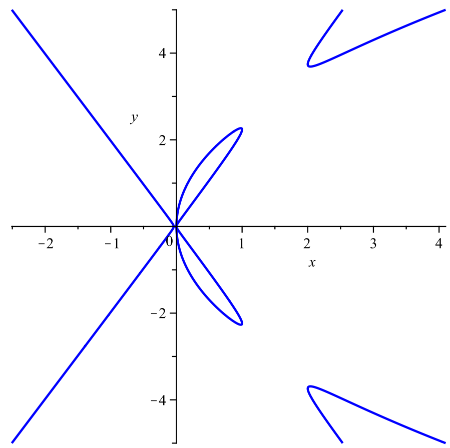


Figure 6: The Ruix-Castizo's curve $(y^2-2x^2-3x)^2-4x^2(1-x)(2-x)=0$.

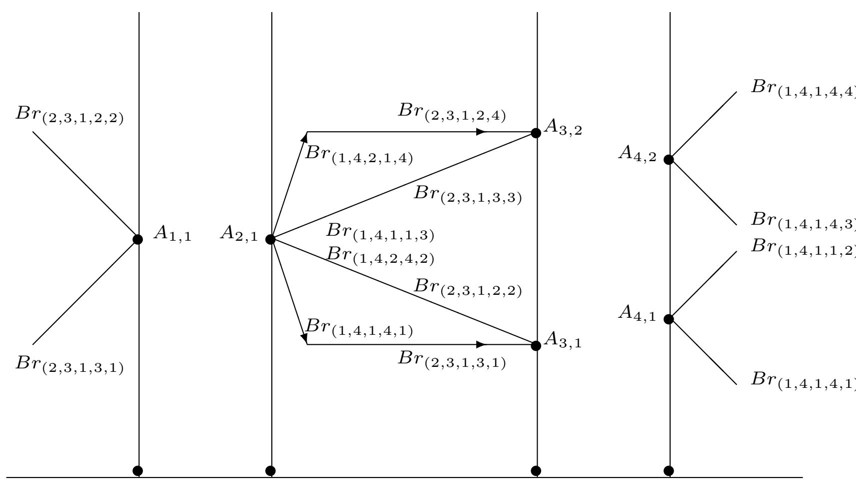


Figure 7

• Algorithm 9: TRREXP

The algorithm says that the second root of discriminant ξ_2 is the projection of the alone singular point of the real curve. So, let $p'(x,y) := p(x+\xi_2, \zeta_{2,1})$ the new polynomial after the translation of $(\xi_2, \zeta_{2,1})$, the point $A_{2,1}$, to the origin. We compute the RREXP($p'(x,y)$):

$$\pi_1 :=< 1, 2, 4, 0, 1, < x^2-6x+1, 2\sqrt{2}+3 >, 1 >$$

$$\pi_2 :=< 1, 2, 4, 0, 1, < x^2-6x+1, -2\sqrt{2}+3 >, 1 >$$

• Algorithm 10: ORDERBr

1) The algorithm computes the local topology over the real roots ξ_1, ξ_3 and ξ_4. The signs of the polynomial $h(x,y)$, (see PHASE 2, 2(a)), at the real points: $A_{1,1} := (\xi_1, \zeta_{1,1})$, $A_{3,1} := (\xi_3, \zeta_{3,1})$, $A_{3,2} := (\xi_3, \zeta_{3,2})$, $A_{4,1} := (\xi_4, \zeta_{4,1})$ and $A_{4,2} := (\xi_4, \zeta_{4,2})$, are respectively: $+1$, $+1$, $+1$, -1 and -1. In this case, we obtain the list of half-branches:

$$\begin{aligned}\Big[\quad &\Big((Br_{(2,3,1,3,1)}, Br_{(2,3,1,2,2)}), A_{1,1}, ()\Big)\\ &\Big((Br_{(2,3,1,3,1)}, Br_{(2,3,1,2,2)}), A_{3,1}, ()\Big)\\ &\Big((Br_{(2,3,1,3,3)}, Br_{(2,3,1,2,4)}), A_{3,2}, ()\Big)\\ &\Big((), A_{4,1}, (Br_{(1,4,1,4,1)}, Br_{(1,4,1,1,2)})\Big)\\ &\Big((), A_{4,2}, (Br_{(1,4,1,4,3)}, Br_{(1,4,1,4,4)})\Big)\quad\Big]\end{aligned}$$

2) The table 1 say that the real branches π_1 and π_2 lie in the quadrants $4 \to 1$. So, we have four real half-branches right of ξ_2: $\pi_{1,4}$, $\pi_{2,4}$, $\pi_{1,1}$ and $\pi_{2,1}$. But $\pi_{1,4}, \pi_{2,4} < \pi_{1,1}, \pi_{2,1}$, by step 2 in the procedure of the ordering. Now, by proposition 4 (1), we have also that, $\pi_{2,4} < \pi_{1,4}$, (because $-2\sqrt{2}+3 < 2\sqrt{2}+3$), and $\pi_{1,1} < \pi_{2,1}$, (because $-2\sqrt{2}+3 < 2\sqrt{2}+3$). Finally, $\pi_{1,4} < \pi_{2,4} < \pi_{1,1} < \pi_{2,1}$. At the point $A_{2,1}$, we obtain this list of half-branches:

$$\Big[\quad\Big((),\quad A_{2,1},\quad (Br_{(1,4,1,4,1)}, Br_{(1,4,2,4,2)},\quad Br_{(1,4,1,1,3)}, Br_{(1,4,2,1,4)})\Big)\quad\Big]$$

• Algorithm 11: TRANSCLOSURE

1) As we can see from figure 7, the real algebraic curve has 4 analytic components:

1. $\Big(Br_{(2,3,1,3,1)}, A_{1,1}, Br_{(2,3,1,2,2)}\Big)$

2. $\Big(Br_{(1,4,1,4,1)}, A_{4,1}, Br_{(1,4,1,1,2)}\Big)$

3. $\Big(Br_{(1,4,1,4,3)}, A_{4,2}, Br_{(1,4,1,4,4)}\Big)$

4. $\Big(Br_{(1,4,1,4,1)}, Br_{(2,3,1,3,1)}, A_{3,1}, Br_{(2,3,1,2,2)}, Br_{(1,4,2,4,2)}, A_{2,1},$
$Br_{(1,4,2,1,4)}, Br_{(2,3,1,2,4)}, A_{3,2}, Br_{(2,3,1,3,3)}, Br_{(1,4,1,1,3)}\Big)$

3.2.4 The final algorithm

Algorithm 12 AnalyticComponentsRealCurve – abbrev. ACRC

input : *The polynomial* $f(x,y)$.

output : *A list of one-dimensional analytic components of the real curve.*

– DESCRIPTION

- **step 1 :** Compute TRREXP with input the polynomial $f(x, y)$, see algorithm 8.
- **step 2 :** Compute ORDERBr with input the result of the previous algorithm, see algorithm 10.
- **step 3 :** Compute TRANSCLOSURE with input the result of the previous algorithm, see algorithm 11.

Remark 3 *The above algorithm gives also a graph homeomorphic to the set:* $\{(a, b) \in \widetilde{R}^2 / f(a, b) = 0\}$.

4 The implementation

The terminology used throughout this section will be AXIOM based, but any type or object oriented language could be used to describe the process.

4.1 The Real Closure

For the determination of RRP-expansions we must utilize an implementation of finite extensions of an ordered field, but the implementations of explicit towers of extensions are not efficient, see [L]. In [LRR] we proposed a new way to implement the real closure of an ordered field which is both generic and efficient. So from a computer algebra point of view the main algorithm ACRC, can be generically implemented. Basic references are: [L] and [LRR].

The domain `RealClosure` is the implementation in AXIOM of the mathematical quantities of the real closure. This domain is the common implementation for the two techniques, interval and Thom's coding, to code a single real algebraic number. The only argument is an object of the type `Field` in AXIOM.

4.2 The algorithms RREXP and RRPSQ

The packages of the two algorithms RREXP and RRPSQ, are respectively:

`RealRationalExpansionsPackage` , abbrev. (`RREP`)

and

`RealRationalPuiseuxSeriesQPackage` , abbrev. (`RRPSP`)

In the first package the main function `realRatExp` takes as argument a polynomial $p(x, y)$ in the domain `RealClosure`, such that $p(0, 0) = 0$, and gives the set of real branches, i.e. the set of $R-\mathcal{RRD}$s (see def. 9), $\{\pi_1, \ldots, \pi_s\}$, such that the real root of each last term τ_{k_i} of π_i, $1 \leq i \leq s$, is simple.

In the second package, the main function `realRatExpSer` take as argument the polynomial $p(x, y)$, and an integer n, and gives the set of real branches $\{\pi_1, \ldots, \pi_s\}$ as previously. But now, each π_s has exactly n $R - \mathcal{RT}$s.

In the following examples we indicate the CPU computing times in secs the main functions `realRatExp` and `realRatExpSer`. Comparisons are not possible because relative programmes, by our knowledge, are not given. The computations have been processed with the method of interval coding using R. Rioboo's implementation, see [LRR]. All of these examples have been tested in an IBM RISC/6000 of the LITP Université P. et M. Curie in Paris.

polynomial	CPU/sec realRatExp	CPU/sec realRatExpSer	♯terms
$y^2+x^3-x^2$ double point	0.5	0.7	5
y^3-x^3 cusp	0.06	0.04	5
$(x^2+y^2-1)(y-x)$ circumference and line	0.02	0.06	5
$y^3-2x^2y^2+3xy-x+y$ random curve	0.01	0.18	5
x^3+y^3-3xy Descartes's folium	0.01	0.62	5
$(x^2+y^2)^2-(2x^2+3y^2)$ lemniscate	0.03	0.5	5
$(x^2+y^2)^2-(x^2-y^2)$ Bernouli's lemniscate	1.01	1.35	5
$(x^2+y^2)^2-4x(x^2-y^2)$ trifolium	0.05	2.02	5
$y^4-2y^3+y^2-3x^2y+2x^4$ tacnode	0.1	0.84	5
$(y^2-2x^2-3x)^2-4x^2(1-x)(2-x)$ Ruiz Castizo's quatric	0.05	10.26	5
$x^4+x^2y^2-2x^2y-xy^2+y^2$ ramphoid cusp	0.49	0.48	5
$y^6+3y^4x^2+3y^2x^4-4y^2x^2+x^6$ ordinary triple point	0.08	7.51	5
$(x^2+y^2-1)^2(x^2-y^2)+xy$ Traverso-Cellini-Gianni	0.07	10.54	5

4.3 The algorithm ACRC

The main algorithm ACRC is the package

`AnalyticComponentsPackage` (abbrev. `ACP`) with arguments : an object R of type Field, which is the field of the coefficients of the defining polynomial of the curve and a boolean which takes the value true for the Thom's coding method and false for the interval coding method. For the moment, the algorithm with the Thom's coding method, compute also the analytic components of the perturbed curves, the curve definined by the polynomials of the general form $p(x, y) \pm \epsilon x^n y^m$, where ϵ an positive infinitesimal, see figure 10 and figure 11, but this version of the ACRC is not efficient. On the contrary, the version with the interval coding method is more efficient but not so generic.

The basic functions are the following :

- TopologicalTypeOf : UP(y,UP(x,R)) $\rightarrow$ List List Record (inPoint:Symbol,Left:Symbol,Right:Symbol)
- AnalyticComponentsOf : UP(y,UP(x,R)) $\rightarrow$ List List Symbol

where : UP abbreviates the UnivariatePolynomial package, the inPoint is the branch-point $A_{i,j}$ (see 1 in 2.2.2) and the Symbol in Left and Right has the form $\mathtt{Br}^{i_4,i_5}_{i_1,i_2,i_3}$, as the half-branch in 2 section 2.2.2.

Examples : The computations of the following examples have been processed with the method of interval coding using R. Rioboo's implementation, see [LRR]. Let $p := (y^2 + x^2 - 1)((x - 1)^2 + 4y^2 - 4)$ be a polynomial in RealClosure(Fraction(Integer)).

1) Let $q := p + 2^{-2}$, see figure 8a. The function TopologicalTypeOf(q) gives:

$[\ [[\mathtt{inPoint}=A_{1,1},\mathtt{Left}=[\,],\mathtt{Right}=[\mathtt{Br}^{4,1}_{1,4,1},\mathtt{Br}^{1,2}_{1,4,1}]],$

$[\mathtt{inPoint}=A_{1,2},\mathtt{Left}=[\,],\mathtt{Right}=[\mathtt{Br}^{4,3}_{1,4,1},\mathtt{Br}^{1,4}_{1,4,1}]]]$

,

$[[\mathtt{inPoint}=A_{2,1},\mathtt{Left}=[\mathtt{Br}^{2,1}_{2,4,1}],\mathtt{Right}=[\mathtt{Br}^{4,1}_{2,4,1}]],$

$[\mathtt{inPoint}=A_{2,2},\mathtt{Left}=[\mathtt{Br}^{3,2}_{2,3,1},\mathtt{Br}^{2,3}_{2,3,1}],\mathtt{Right}=[\,]],$

$[\mathtt{inPoint}=A_{2,3},\mathtt{Left}=[\mathtt{Br}^{2,4}_{2,4,1}],\mathtt{Right}=[\mathtt{Br}^{4,2}_{2,4,1}]]]$

,

$[[\mathtt{inPoint}=A_{3,1},\mathtt{Left}=[\mathtt{Br}^{3,1}_{2,3,1},\mathtt{Br}^{2,2}_{2,3,1}],\mathtt{Right}=[\,]]]\ ]$

The function `AnalyticComponentsOf` gives one analytic component of the curve:
$[\mathtt{Br}^{4,1}_{1,4,1}, \mathtt{Br}^{2,1}_{2,4,1}, \mathtt{A}_{2,1}$
$, \mathtt{Br}^{4,1}_{2,4,1}, \mathtt{Br}^{3,1}_{2,3,1}, \mathtt{A}_{3,1}, \mathtt{Br}^{2,2}_{2,3,1}, \mathtt{Br}^{4,2}_{2,4,1},$
$\mathtt{A}_{2,3}, \mathtt{Br}^{2,4}_{2,4,1}, \mathtt{Br}^{1,4}_{1,4,1}, \mathtt{A}_{1,2}, \mathtt{Br}^{4,3}_{1,4,1}, \mathtt{Br}^{2,3}_{2,3,1}, \mathtt{A}_{2,2}$
$, \mathtt{Br}^{3,2}_{2,3,1},$
$\mathtt{Br}^{1,2}_{1,4,1}, \mathtt{A}_{1,1}]$

In the following diagrams the adjacent real half-branches in a real point are illustrated by the arrowheads, if it necesary.

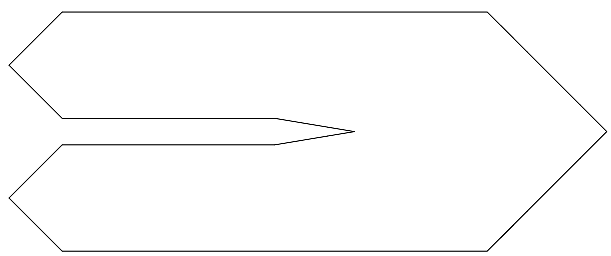


Figure 8a: 1 anal. component

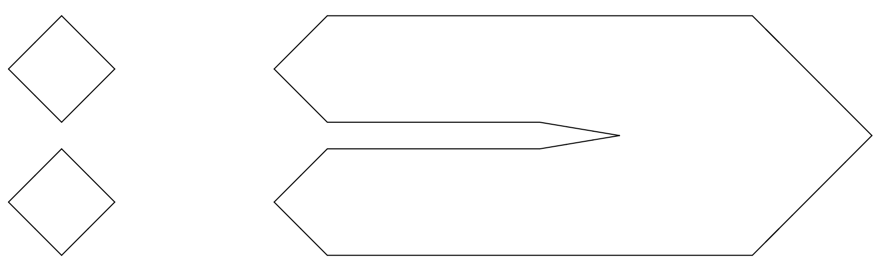


Figure 8b: 3 anal. components

2) If $q := p + 2^{-4}$, the corresponding curve has 3 analytic components as we can see in figure 8b.

3) Let $p := y^6 - 12y^4 + 36y^2 + x^6 - 11x^4 + 35x^2 - 57$ be Santos's curve; see also 11 below. This curve has one analytic component as we can see following the arrowheads given by our local topological analysis in figure 9.

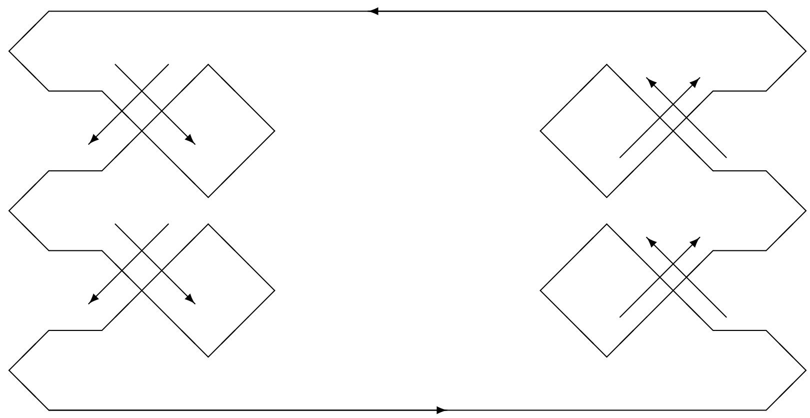


Figure 9: 1 anal. component

4) Let $p(x,y) := (x^2+y^2-1)(y-x-1)$, be the circumference and line see 1 below. If ϵ is an positive infinitesimal, then the version of `ACRC` with the Thom's coding method gives the following shape with two analytic components for the curve $p(x,y)+\epsilon$, figure 10,

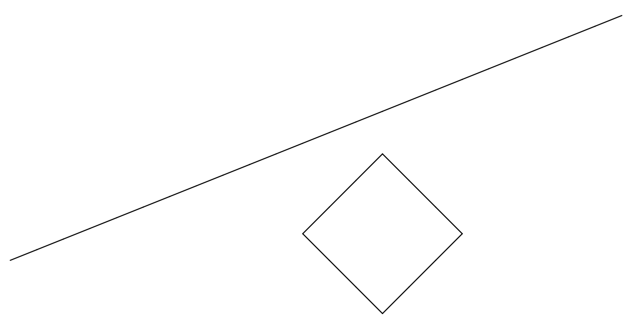


Figure 10: 2 anal. components

and two analytic components for the curve $p(x,y)-\epsilon$, figure 11.

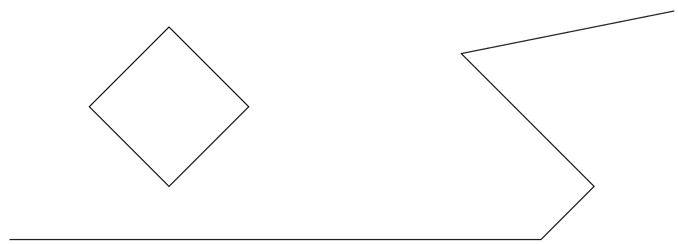
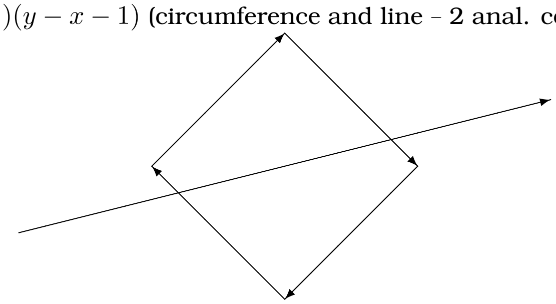


Figure 11: 2 anal. components

In the following examples we show the defining polynomials, the corresponding shapes taken by our algorithm, and the CPU computing times in secs. The first one is the time given by the algorithm in [CP3R], say the `ACP`, and the second by our implementation `ACRC`. All of the examples have been tested in an IBM RISC/6000 of the LITP Université P. et M. Curie in Paris.

1: $(x^2+y^2-1)(y-x-1)$ (circumference and line – 2 anal. components)

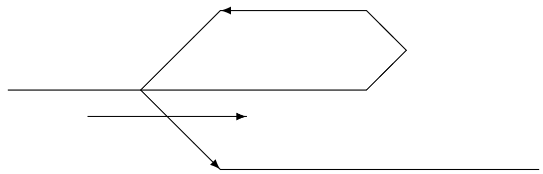


Computing time : `ACP` : 1 , `ACRC` : 3.4

2: $y^3-2x^2y^2+3xy-x+y$ (random curve – 3 anal. components)

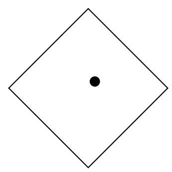
Computing time : `ACP` : 526 , `ACRC` : 136

3: x^3+y^3-3xy (Descartes's folium – 1 anal. component)



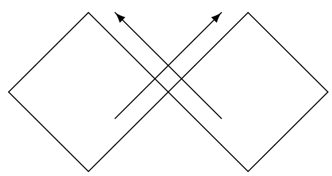
Computing time : `ACP` : 2.8 , `ACRC` : 3

4: $(x^2+y^2)^2-(2x^2+3y^2)$ (lemniscate – 2 anal. components)



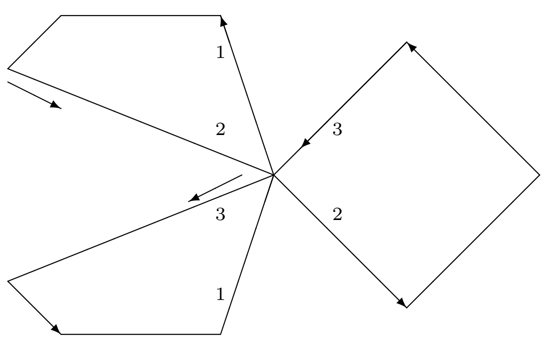
Computing time : `ACP` : 0.1 , `ACRC` : 4

5: $(x^2+y^2)^2-(x^2-y^2)$ (Bernouli's lemniscate – 1 anal. component)



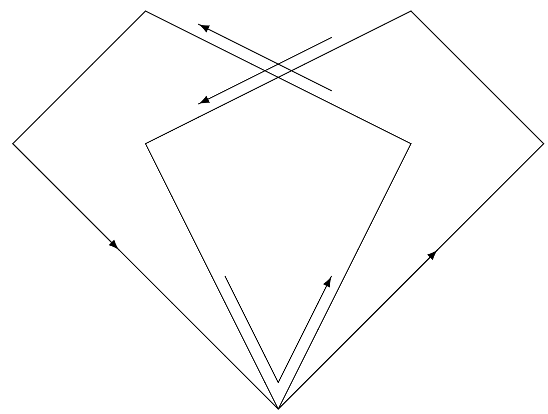
Computing time : `ACP` : 0.67 , `ACRC` 2.5

6: $(x^2+y^2)^2-4x(x^2-y^2)$ (trifolium – 1 anal. component)



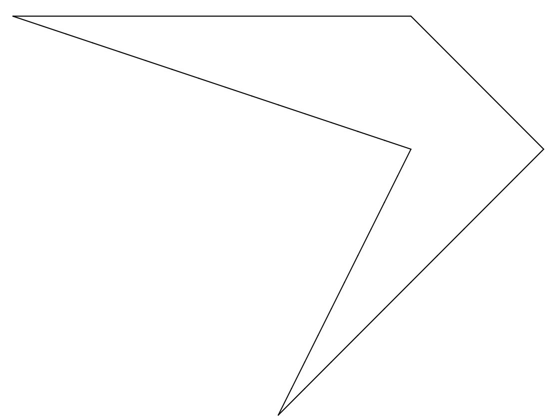
Computing time : `ACP` : 1 , `ACRC` : 3

7: $y^4 - 2y^3 + y^2 - 3x^2y + 2x^4$ (tacnode – 1 anal. component)



Computing time : `ACP` : $>$ 2000 , `ACRC` : 185

8: $x^4+x^2y^2-2x^2y-xy^2+y^2 = 2y^4+(6x-3)y^3+(7x^2-5x+1)y^2+(4x^3-2x^2)y+x^4$ (ramphoid cusp – 1 anal. component)



Computing time : `ACP` : $>$ 2000 , `ACRC` : 525

9: $(y^2 - 2x^2 - 3x)^2 - 4x^2(1-x)(2-x)$ (Ruiz Castizo's quatric), see figure 7.
Computing time : `ACP` : 3.5 , `ACRC` : 5

10: $(y^2 + x^2 - 1)((x-1)^2 + 4y^2 - 4) + 2^{-2}$, (H. Hong), see figure 8a.
Computing time : `ACP` : 348 , `ACRC` : 57

11: $y^6 - 12y^4 + 36y^2 + x^6 - 11x^4 + 35x^2 - 57$, (F. Santos), see figure 9
Computing time : `ACP` : $>$ 2000 , `ACRC` : 162

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