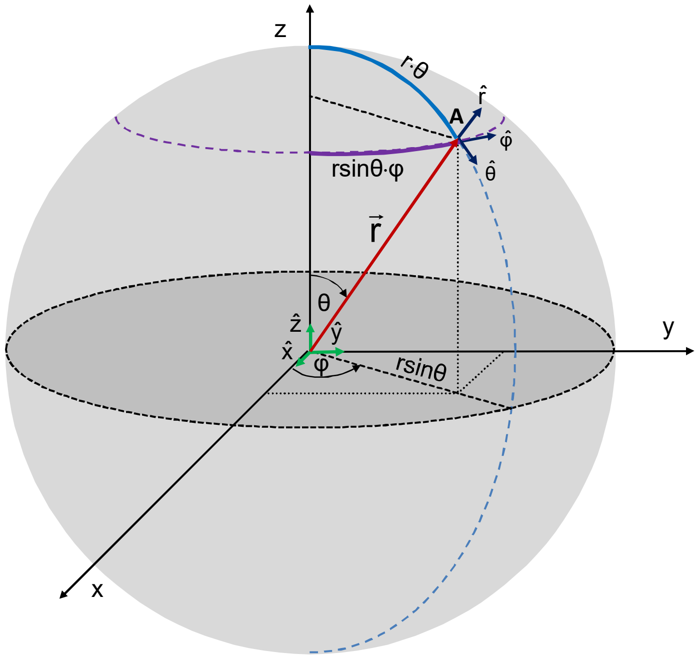
Σφαιρικές συντεταγμένες



Έστω διάνυσμα θέσης ενός σημείου Α $\vec{r} = x \cdot \hat{x} + y \cdot \hat{y} + z \cdot \hat{z}$ σε καρτεσιανές συντεταγμένες όπου $\hat{x}, \hat{y}, \hat{z}$ τα μοναδιαία διανύσματα. Σε σφαιρικές συντεταγμένες το διάνυσμα θέσης του σημείου Α θα έχει συντεταγμένες r, θ, φ και η σχέσεις των καρτεσιανών με τις σφαιρικές συντεταγμένες είναι:

$$x = r\sin\theta\cos\varphi \qquad (1)$$

$$y = r\sin\theta\sin\varphi \qquad (2)$$

$$z = r\cos\theta \qquad (3)$$

επομένως:

$$\vec{r} = r\sin\theta\cos\varphi \cdot \hat{x} + r\sin\theta\sin\varphi \cdot \hat{y} + r\cos\theta \cdot \hat{z} \qquad (4)$$

με $0 \le r < \infty$ η **ακτινική** συντεταγμένη, $0 \le \theta \le \pi$ rad η **πολική** συντεταγμένη και $0 \le \varphi < 2\pi$ rad η **αζιμουθιακή** συντεταγμένη.
Το διάνυσμα θέσης του σημείου Α σε σφαιρικές συντεταγμένες μπορεί να γραφεί ως:

$$\vec{r} = r \cdot \hat{r} + r\theta \cdot \hat{\theta} + r\sin\theta \cdot \varphi \cdot \hat{\varphi} \qquad (5)$$

Στοιχειώδες μήκος: $$d\vec{r} = dr \cdot \hat{r} + r \cdot d\theta \cdot \hat{\theta} + r\sin\theta \cdot d\varphi \cdot \hat{\varphi} \qquad (6)$$

Στοιχειώδης όγκος: $$dV = dx \cdot dy \cdot dz = dr \cdot (r d\theta) \cdot (r\sin\theta d\varphi) = r^2 \sin\theta \cdot dr \cdot d\theta \cdot d\varphi \qquad (7)$$

Η αντίστοιχες σχέσεις των r, φ, θ με τα x, y, z είναι: $$r = \left(x^2 + y^2 + z^2\right)^{1/2} \qquad (8)$$

$$\tan\varphi = \frac{y}{x} \Leftrightarrow \varphi = \tan^{-1}\left(\frac{y}{x}\right) \quad (9) \qquad \text{και} \qquad \cos\theta = \frac{z}{r} \Leftrightarrow \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \quad (10)$$

Κλίση βαθμωτής συνάρτησης σε σφαιρικές συντεταγμένες:

Κανόνες μερικών παραγώγων :

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial y}\right)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial z}\right)$$

Παράγωγοι αντίστροφων τριγωνομετρικών συναρτήσεων

Έστω η συνάρτηση: y=cosx (1)

Η αντίστροφη συνάρτηση είναι η λύση της παραπάνω εξίσωσης, δηλαδή: $x=\cos^{-1}y$ (2)

Η παράγωγος της αντίστροφης συνάρτησης υπολογίζεται ως εξής:

Παραγωγίζουμε τη σχέση (1): $\frac{dy}{dx}=-\sin x \Leftrightarrow \frac{dx}{dy}=\frac{1}{-\sin x}\xrightarrow{(2)}\frac{d(\cos^{-1}y)}{dy}=\frac{1}{-\sqrt{1-\cos^2 x}}\xrightarrow{(1)}$

$$\frac{d(\cos^{-1}y)}{dy}=-\frac{1}{\sqrt{1-y^2}} \qquad 0<\cos^{-1}y<\pi$$

Έστω η συνάρτηση: y=tanx (3)

Η αντίστροφη συνάρτηση είναι η λύση της παραπάνω εξίσωσης, δηλαδή: $x=\tan^{-1}y$ (4)

Η παράγωγος της αντίστροφης συνάρτησης υπολογίζεται ως εξής:

Παραγωγίζουμε τη σχέση (3): $\frac{dy}{dx}=\frac{1}{\cos^2 x}\Leftrightarrow \frac{dx}{dy}=\cos^2 x\xrightarrow{(2)}\frac{d(\tan^{-1}y)}{dy}=\cos^2 x$ (5)

Όμως: $\frac{1}{\cos^2 x}=\frac{\sin^2 x+\cos^2 x}{\cos^2 x}=\tan^2 x+1$ (6)

$$(5)\xrightarrow{(6)}\frac{d(\tan^{-1}y)}{dy}=\frac{1}{\tan^2 x+1}\xrightarrow{(3)}\frac{d(\tan^{-1}y)}{dy}=\frac{1}{y^2+1} \qquad -\pi/2<\tan^{-1}y<\pi/2$$

όπου: $\frac{\partial r}{\partial x}=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}2x\xrightarrow[(8)]{(1)}\frac{\partial r}{\partial x}=\sin\theta\cos\varphi$

$$\frac{\partial r}{\partial y}=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}2y\xrightarrow[(8)]{(1)}\frac{\partial r}{\partial y}=\sin\theta\sin\varphi, \qquad \frac{\partial r}{\partial z}=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}2z\xrightarrow[(8)]{(1)}\frac{\partial r}{\partial z}=\cos\theta$$

$$\frac{\partial\varphi}{\partial x}=\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=\frac{x^2}{y^2+x^2}\left(-\frac{y}{x^2}\right)\xrightarrow{(2)}\frac{\partial\varphi}{\partial x}=-\frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta}=-\frac{\sin\varphi}{r\sin\theta},$$

$$\frac{\partial\varphi}{\partial y}=\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{x^2}{y^2+x^2}\left(\frac{1}{x}\right)\xrightarrow{(1)}\frac{\partial\varphi}{\partial y}=\frac{r\sin\theta\cos\varphi}{r^2\sin^2\theta}=\frac{\cos\varphi}{r\sin\theta},$$

$$\frac{\partial\varphi}{\partial z}=\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial z}\left(\frac{y}{x}\right)=0$$

$$\frac{\partial\theta}{\partial x}=-\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\partial}{\partial x}\frac{z}{\sqrt{x^2+y^2+z^2}}\xrightarrow[(8)]{(3)}\frac{\partial\theta}{\partial x}=-\frac{1}{\sqrt{1-\frac{r^2\cos^2\theta}{r^2}}}\left(-\frac{1}{2}\right)\frac{2xz}{\left(x^2+y^2+z^2\right)^{3/2}}\Leftrightarrow$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{\sin\theta}\frac{r^2\sin\theta\cos\varphi\cos\theta}{r^3}\Leftrightarrow\frac{\partial\theta}{\partial x}=\frac{\cos\varphi\cos\theta}{r}$$

$$\frac{\partial\theta}{\partial y}=-\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\partial}{\partial y}\frac{z}{\sqrt{x^2+y^2+z^2}}\xrightarrow[(8)]{(3)}\frac{\partial\theta}{\partial y}=-\frac{1}{\sin\theta}\left(-\frac{1}{2}\right)\frac{2yz}{\left(x^2+y^2+z^2\right)^{3/2}}\Leftrightarrow$$

$$\frac{\partial\theta}{\partial y}=\frac{1}{\sin\theta}\frac{r^2\sin\theta\sin\varphi\cos\theta}{r^3}\Leftrightarrow\frac{\partial\theta}{\partial y}=\frac{\sin\varphi\cos\theta}{r}$$

$$\frac{\partial\theta}{\partial z}=-\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\partial}{\partial z}\frac{z}{\sqrt{x^2+y^2+z^2}}\xrightarrow[(8)]{(3)}\frac{\partial\theta}{\partial z}=-\frac{1}{\sin\theta}\left[\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}-\frac{1}{2}\frac{2z^2}{\left(x^2+y^2+z^2\right)^{3/2}}\right]\Leftrightarrow$$

$$\frac{\partial\theta}{\partial z}=-\frac{1}{\sin\theta}\left(\frac{1}{r}-\frac{r^2\cos^2\theta}{r^3}\right)\Leftrightarrow\frac{\partial\theta}{\partial z}=-\frac{1}{\sin\theta}\frac{\sin^2\theta}{r}\Leftrightarrow\frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{r}$$

Για τα μοναδιαία διανύσματα ισχύει: $\hat{r}=\dfrac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}=\dfrac{\vec{r}}{|\vec{r}|}$, $\hat{\varphi}=\dfrac{\frac{\partial\vec{r}}{\partial\varphi}}{\left|\frac{\partial\vec{r}}{\partial\varphi}\right|}$ και $\hat{\theta}=\dfrac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}$

$$|\vec{r}|=\sqrt{(r\sin\theta\cos\varphi)^2+(r\sin\theta\sin\varphi)^2+(r\cos\theta)^2}=r\sqrt{\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+\cos^2\theta}=r$$

Οπότε: $$\boxed{\hat{r}=\sin\theta\cos\varphi\cdot\hat{x}+\sin\theta\sin\varphi\cdot\hat{y}+\cos\theta\cdot\hat{z}}\qquad(11)$$

$$\frac{\partial\vec{r}}{\partial\varphi}=-r\sin\theta\sin\varphi\cdot\hat{x}+r\sin\theta\cos\varphi\cdot\hat{y}$$ και

$$\left|\frac{\partial\vec{r}}{\partial\varphi}\right|=\sqrt{(-r\sin\theta\sin\varphi)^2+(r\sin\theta\cos\varphi)^2}=r\sqrt{\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)}=r\sin\theta$$

Οπότε: $$\boxed{\hat{\varphi}=-\sin\varphi\cdot\hat{x}+\cos\varphi\cdot\hat{y}}\qquad(12)$$

$$\frac{\partial\vec{r}}{\partial\theta}=r\cos\theta\cos\varphi\cdot\hat{x}+r\cos\theta\sin\varphi\cdot\hat{y}-r\sin\theta\cdot\hat{z}$$ και

$$\left|\frac{\partial\vec{r}}{\partial\theta}\right|=\sqrt{(r\cos\theta\cos\varphi)^2+(r\cos\theta\sin\varphi)^2+(-\sin\theta)^2}=r\sqrt{\cos^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+\sin^2\theta}=r$$

Οπότε: $$\boxed{\hat{\theta}=\cos\theta\cos\varphi\cdot\hat{x}+\cos\theta\sin\varphi\cdot\hat{y}-\sin\theta\cdot\hat{z}}\qquad(13)$$

Αντίστοιχα για τα μοναδιαία: $\hat{x} = \dfrac{\frac{\partial \vec{r}}{\partial x}}{\left|\frac{\partial \vec{r}}{\partial x}\right|}$, $\hat{y} = \dfrac{\frac{\partial \vec{r}}{\partial y}}{\left|\frac{\partial \vec{r}}{\partial y}\right|}$ και $\hat{z} = \dfrac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|}$

$$\frac{\partial \vec{r}}{\partial x} = \frac{\partial r}{\partial x}\hat{r} + r\frac{\partial \theta}{\partial x}\hat{\theta} + r\sin\varphi \cdot \frac{\partial \varphi}{\partial x}\hat{\varphi} \Leftrightarrow \frac{\partial \vec{r}}{\partial x} = \sin\theta\cos\varphi \cdot \hat{r} + \cos\varphi\cos\theta \cdot \hat{\theta} - \sin\varphi \cdot \hat{\varphi} \quad \text{και}$$

$$\left|\frac{\partial \vec{r}}{\partial x}\right| = \sqrt{(\sin\theta\cos\varphi)^2 + (\cos\varphi\cos\theta)^2 + (-\sin\varphi)^2} = 1$$

Οπότε: $\boxed{\hat{x} = \sin\theta\cos\varphi \cdot \hat{r} + \cos\varphi\cos\theta \cdot \hat{\theta} - \sin\varphi \cdot \hat{\varphi}}$ (14)

$$\frac{\partial \vec{r}}{\partial y} = \frac{\partial r}{\partial y}\hat{r} + r\frac{\partial \theta}{\partial y}\hat{\theta} + r\sin\varphi \cdot \frac{\partial \varphi}{\partial y}\hat{\varphi} \Leftrightarrow \frac{\partial \vec{r}}{\partial y} = \sin\theta\sin\varphi \cdot \hat{r} + \sin\varphi\cos\theta \cdot \hat{\theta} + \cos\varphi \cdot \hat{\varphi} \quad \text{και}$$

$$\left|\frac{\partial \vec{r}}{\partial y}\right| = \sqrt{(\sin\theta\sin\varphi)^2 + (\sin\varphi\cos\theta)^2 + (\cos\varphi)^2} = 1$$

Οπότε: $\boxed{\hat{y} = \sin\theta\sin\varphi \cdot \hat{r} + \sin\varphi\cos\theta \cdot \hat{\theta} + \cos\varphi \cdot \hat{\varphi}}$ (15)

$$\frac{\partial \vec{r}}{\partial z} = \frac{\partial r}{\partial z}\hat{r} + r\frac{\partial \theta}{\partial z}\hat{\theta} + r\sin\varphi \cdot \frac{\partial \varphi}{\partial z}\hat{\varphi} \Leftrightarrow \frac{\partial \vec{r}}{\partial z} = \cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta} \quad \text{και}$$

$$\left|\frac{\partial \vec{r}}{\partial z}\right| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

Οπότε: $\boxed{\hat{z} = \cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta}}$ (16)

- Μερικές παράγωγοι των μοναδιαίων διανυσμάτων:

$$\frac{\partial \hat{r}}{\partial r} = 0, \qquad \frac{\partial \hat{r}}{\partial \varphi} = -\sin\theta\sin\varphi \cdot \hat{x} + \sin\theta\cos\varphi \cdot \hat{y} = \sin\theta \cdot \hat{\varphi},$$

$$\frac{\partial \hat{r}}{\partial \theta} = \cos\theta\cos\varphi \cdot \hat{x} + \cos\theta\sin\varphi \cdot \hat{y} - \sin\theta \cdot \hat{z} = \hat{\theta} \tag{17}$$

$$\frac{\partial \hat{\varphi}}{\partial r} = 0, \qquad \frac{\partial \hat{\varphi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{\varphi}}{\partial \varphi} = -\cos\varphi \cdot \hat{x} - \sin\varphi \cdot \hat{y} = -\sin\theta \cdot \hat{r} - \cos\theta \cdot \hat{\theta} \tag{18}$$

όπου: $-\sin\theta \cdot \hat{r} = -\sin^2\theta\cos\varphi \cdot \hat{x} - \sin^2\theta\sin\varphi \cdot \hat{y} - \sin\theta\cos\theta \cdot \hat{z}$

και

$$-\cos\theta\cdot\hat{\theta} = -\cos^2\theta\cos\varphi\cdot\hat{x} - \cos^2\theta\sin\varphi\cdot\hat{y} + \cos\theta\sin\theta\cdot\hat{z}$$

Οπότε:

$$-\sin\theta\cdot\hat{r} - \cos\theta\cdot\hat{\theta} = -\left(\cos^2\theta + \sin^2\theta\right)\cos\varphi\cdot\hat{x} - \left(\cos^2\theta + \sin^2\theta\right)\sin\varphi\cdot\hat{y} - \cos\theta\sin\theta\cdot\hat{z} + \sin\theta\cos\theta\cdot\hat{z} =$$

$$= -\cos\varphi\cdot\hat{x} - \sin\varphi\cdot\hat{y}$$

$$\frac{\partial\hat{\theta}}{\partial r} = 0, \qquad \frac{\partial\hat{\theta}}{\partial\varphi} = -\cos\theta\sin\varphi\cdot\hat{x} + \cos\theta\cos\varphi\cdot\hat{y} = \cos\theta\cdot\hat{\varphi},$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = -\sin\theta\cos\varphi\cdot\hat{x} - \sin\theta\sin\varphi\cdot\hat{y} - \cos\theta\cdot\hat{z} = -\hat{r} \qquad (19)$$

Η **κλίση** βαθμωτής συνάρτησης **f** σε σφαιρικές συντεταγμένες γράφεται:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \Leftrightarrow \vec{\nabla} f = \left[\frac{\partial f}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial f}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial f}{\partial\theta}\left(\frac{\partial\theta}{\partial x}\right)\right]\left(\sin\theta\cos\varphi\cdot\hat{r} + \cos\varphi\cos\theta\cdot\hat{\theta} - \sin\varphi\cdot\hat{\varphi}\right) +$$

$$+\left[\frac{\partial f}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial f}{\partial\varphi}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial f}{\partial\theta}\left(\frac{\partial\theta}{\partial y}\right)\right]\left(\sin\theta\sin\varphi\cdot\hat{r} + \sin\varphi\cos\theta\cdot\hat{\theta} + \cos\varphi\cdot\hat{\varphi}\right) +$$

$$+\left[\frac{\partial f}{\partial r}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial f}{\partial\varphi}\left(\frac{\partial\varphi}{\partial z}\right) + \frac{\partial f}{\partial\theta}\left(\frac{\partial\theta}{\partial z}\right)\right]\left(\cos\theta\cdot\hat{r} - \sin\theta\cdot\hat{\theta}\right) \Leftrightarrow$$

$$\vec{\nabla} f = \left(\sin\theta\cos\varphi\frac{\partial f}{\partial r} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi} + \frac{\cos\varphi\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)\left(\sin\theta\cos\varphi\cdot\hat{r} + \cos\varphi\cos\theta\cdot\hat{\theta} - \sin\varphi\cdot\hat{\varphi}\right) +$$

$$+\left(\sin\theta\sin\varphi\frac{\partial f}{\partial r} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi} + \frac{\sin\varphi\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)\left(\sin\theta\sin\varphi\cdot\hat{r} + \sin\varphi\cos\theta\cdot\hat{\theta} + \cos\varphi\cdot\hat{\varphi}\right) +$$

$$+\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)\left(\cos\theta\cdot\hat{r} - \sin\theta\cdot\hat{\theta}\right) \Leftrightarrow$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta\right)\cdot\hat{r} + \frac{\partial f}{\partial r}\left(\sin\theta\cos^2\varphi\cos\theta + \sin\theta\sin^2\varphi\cos\theta - \cos\theta\sin\theta\right)\cdot\hat{\theta} +$$

$$\frac{\partial f}{\partial r}\left(-\sin\varphi\sin\theta\cos\varphi + \sin\theta\sin\varphi\cos\varphi\right)\cdot\hat{\varphi} + \frac{\partial f}{\partial\varphi}\left(-\frac{\sin\varphi}{r\sin\theta}\sin\theta\cos\varphi + \frac{\cos\varphi}{r\sin\theta}\sin\theta\sin\varphi\right)\cdot\hat{r} +$$

$$\frac{\partial f}{\partial\varphi}\left(-\frac{\sin\varphi}{r\sin\theta}\cos\varphi\cos\theta + \frac{\cos\varphi}{r\sin\theta}\sin\varphi\cos\theta\right)\cdot\hat{\theta} + \frac{\partial f}{\partial\varphi}\left(\frac{\sin\varphi}{r\sin\theta}\sin\varphi + \frac{\cos\varphi}{r\sin\theta}\cos\varphi\right)\cdot\hat{\varphi} +$$

$$\frac{\partial f}{\partial\theta}\left(\frac{\cos^2\varphi\cos\theta\sin\theta}{r} + \frac{\sin^2\varphi\cos\theta\sin\theta}{r} - \frac{\sin\theta\cos\theta}{r}\right)\cdot\hat{r} + \frac{\partial f}{\partial\theta}\left(\frac{\cos^2\varphi\cos^2\theta}{r} + \frac{\sin^2\varphi\cos^2\theta}{r} + \frac{\sin^2\theta}{r}\right)\cdot\hat{\theta} +$$

$$+\frac{\partial f}{\partial\theta}\left(-\frac{\cos\varphi\cos\theta}{r}\sin\varphi + \frac{\sin\varphi\cos\theta}{r}\cos\varphi\right)\cdot\hat{\varphi} \Leftrightarrow \boxed{\vec{\nabla} f = \hat{r}\cdot\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial f}{\partial\varphi}} \qquad (20)$$

Η **απόκλιση** μιας διανυσματικής συνάρτησης $\vec{F}(r,\theta,\varphi) = F_r \cdot \hat{r} + F_\theta \cdot \hat{\theta} + F_\varphi \cdot \hat{\varphi}$ σε σφαιρικές συντεταγμένες είναι:

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\left(F_r \cdot \hat{r} + F_\theta \cdot \hat{\theta} + F_\varphi \cdot \hat{\varphi}\right) \Leftrightarrow$$

$$\vec{\nabla} \cdot \vec{F} = \hat{r}\frac{\partial F_r}{\partial r}\cdot\hat{r} + \hat{r}\cdot F_r\frac{\partial \hat{r}}{\partial r} + \hat{r}\frac{\partial F_\theta}{\partial r}\cdot\hat{\theta} + \hat{r}\cdot F_\theta\frac{\partial \hat{\theta}}{\partial r} + \hat{r}\frac{\partial F_\varphi}{\partial r}\cdot\hat{\varphi} + \hat{r}\cdot F_\varphi\frac{\partial \hat{\varphi}}{\partial r} +$$

$$+\frac{\hat{\theta}}{r}\frac{\partial F_r}{\partial \theta}\hat{r} + \frac{F_r\hat{\theta}}{r}\frac{\partial \hat{r}}{\partial \theta} + \frac{\hat{\theta}}{r}\frac{\partial F_\theta}{\partial \theta}\hat{\theta} + \frac{F_\theta\hat{\theta}}{r}\frac{\partial \hat{\theta}}{\partial \theta} + \frac{\hat{\theta}}{r}\frac{\partial F_\varphi}{\partial \theta}\hat{\varphi} + \frac{F_\varphi\hat{\theta}}{r}\frac{\partial \hat{\varphi}}{\partial \theta} +$$

$$+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial F_r}{\partial \varphi}\hat{r} + \frac{F_r\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{r}}{\partial \varphi} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial F_\theta}{\partial \varphi}\hat{\theta} + \frac{F_\theta\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{\theta}}{\partial \varphi} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}\hat{\varphi} + \frac{F_\varphi\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{\varphi}}{\partial \varphi} \Leftrightarrow$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_r}{\partial r} + \frac{F_r\hat{\theta}}{r}\frac{\partial \hat{r}}{\partial \theta} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{F_\theta\hat{\theta}}{r}\frac{\partial \hat{\theta}}{\partial \theta} + \frac{F_\varphi\hat{\theta}}{r}\frac{\partial \hat{\varphi}}{\partial \theta} + \frac{F_r\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{r}}{\partial \varphi} + \frac{F_\theta\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{\theta}}{\partial \varphi} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} + \frac{F_\varphi\cdot\hat{\varphi}}{r\sin\theta}\frac{\partial \hat{\varphi}}{\partial \varphi} \xrightarrow[(19)]{(17),(18)}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_r}{\partial r} + \frac{F_r\hat{\theta}}{r}\hat{\theta} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{F_\theta\hat{\theta}}{r}\left(-\hat{r}\right) + \frac{F_r\cdot\hat{\varphi}}{r}\hat{\varphi} + \frac{F_\theta\cos\theta\cdot\hat{\varphi}}{r\sin\theta}\hat{\varphi} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} + \frac{F_\varphi\cdot\hat{\varphi}}{r\sin\theta}\left(-\sin\theta\cdot\hat{r} - \cos\cdot\hat{\theta}\right) \Leftrightarrow$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_r}{\partial r} + \frac{2F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{F_\theta\cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} \Leftrightarrow$$

$$\boxed{\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2}\frac{\partial\left(r^2F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(F_\theta\sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}} \qquad (21)$$

Ο **στροβιλισμός** μιας διανυσματικής συνάρτησης $\vec{F}(r,\theta,\varphi) = F_r \cdot \hat{r} + F_\theta \cdot \hat{\theta} + F_\varphi \cdot \hat{\varphi}$ σε σφαιρικές συντεταγμένες είναι:

$$\vec{\nabla} \times \vec{F} = \left(\hat{r}\cdot\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\times\left(F_r \cdot \hat{r} + F_\theta \cdot \hat{\theta} + F_\varphi \cdot \hat{\varphi}\right) \Leftrightarrow$$

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_r}{\partial r}\hat{r}\times\hat{r} + \hat{r}\times\frac{\partial \hat{r}}{\partial r}F_r + \frac{\partial F_\theta}{\partial r}\hat{r}\times\hat{\theta} + \hat{r}\times\frac{\partial \hat{\theta}}{\partial r}F_\theta + \frac{\partial F_\varphi}{\partial r}\hat{r}\times\hat{\varphi} + \hat{r}\times\frac{\partial \hat{\varphi}}{\partial r}F_\varphi +$$

$$+\frac{1}{r}\frac{\partial F_r}{\partial \theta}\hat{\theta}\times\hat{r} + \frac{F_r}{r}\hat{\theta}\times\frac{\partial \hat{r}}{\partial \theta} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\hat{\theta}\times\hat{\theta} + \frac{F_\theta}{r}\hat{\theta}\times\frac{\partial \hat{\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial F_\varphi}{\partial \theta}\hat{\theta}\times\hat{\varphi} + \frac{F_\varphi}{r}\hat{\theta}\times\frac{\partial \hat{\varphi}}{\partial \theta} +$$

$$+\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \varphi}\hat{\varphi}\times\hat{r} + \frac{F_r}{r\sin\theta}\hat{\varphi}\times\frac{\partial \hat{r}}{\partial \varphi} + \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial \varphi}\hat{\varphi}\times\hat{\theta} + \frac{F_\theta}{r\sin\theta}\hat{\varphi}\times\frac{\partial \hat{\theta}}{\partial \varphi} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}\hat{\varphi}\times\hat{\varphi} + \frac{F_\varphi}{r\sin\theta}\hat{\varphi}\times\frac{\partial \hat{\varphi}}{\partial \varphi} \xrightarrow[(19)]{(17),(18)}$$

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_\theta}{\partial r}\hat{\varphi} + \frac{\partial F_\varphi}{\partial r}(-\hat{\theta}) + \frac{1}{r}\frac{\partial F_r}{\partial \theta}(-\hat{\varphi}) + \frac{F_r}{r}\hat{\theta}\times\hat{\theta} + \frac{F_\theta}{r}\hat{\theta}\times(-\hat{r}) + \frac{1}{r}\frac{\partial F_\varphi}{\partial \theta}\hat{r} +$$

$$+\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\varphi}\hat{\theta}+\frac{F_r}{r\sin\theta}\hat{\varphi}\times(\sin\theta\cdot\hat{\varphi})+\frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\varphi}(-\hat{r})+\frac{F_\theta}{r\sin\theta}\hat{\varphi}\times(\cos\theta\cdot\hat{\varphi})+\frac{F_\varphi}{r\sin\theta}\hat{\varphi}\times(-\sin\theta\cdot\hat{r}-\cos\theta\cdot\hat{\theta})\Leftrightarrow$$

$$\vec{\nabla}\times\vec{F}=\frac{\partial F_\theta}{\partial r}\hat{\varphi}-\frac{\partial F_\varphi}{\partial r}\hat{\theta}-\frac{1}{r}\frac{\partial F_r}{\partial\theta}\hat{\varphi}+\frac{F_\theta}{r}\hat{\varphi}+\frac{1}{r}\frac{\partial F_\varphi}{\partial\theta}\hat{r}+\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\varphi}\hat{\theta}-\frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\varphi}\hat{r}-\frac{F_\varphi}{r}\hat{\theta}+\frac{\cos\theta F_\varphi}{r\sin\theta}\hat{r}\Leftrightarrow$$

$$\vec{\nabla}\times\vec{F}=\hat{r}\left(\frac{1}{r}\frac{\partial F_\varphi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\varphi}+\frac{\cos\theta\cdot F_\varphi}{r\sin\theta}\right)+\hat{\theta}\left(-\frac{\partial F_\varphi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{F_\varphi}{r}\right)+\hat{\varphi}\left(\frac{\partial F_\theta}{\partial r}-\frac{1}{r}\frac{\partial F_r}{\partial\theta}+\frac{F_\theta}{r}\right)\Leftrightarrow$$

$$\boxed{\vec{\nabla}\times\vec{F}=\frac{\hat{r}}{r\sin\theta}\left[\frac{\partial(\sin\theta\cdot F_\varphi)}{\partial\theta}-\frac{\partial F_\theta}{\partial\varphi}\right]+\frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{\partial(r\cdot F_\varphi)}{\partial r}\right]+\frac{\hat{\varphi}}{r}\left[\frac{\partial(r\cdot F_\theta)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right]}\qquad(22)$$

Η **Λαπλασιανή** μιας βαθμωτής **f** συνάρτησης σε σφαιρικές συντεταγμένες είναι:

$$\vec{\nabla}^2 f=\vec{\nabla}\cdot\vec{\nabla}f=\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\frac{\partial f}{\partial r}\cdot\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\cdot\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\cdot\hat{\varphi}\right)\Leftrightarrow$$

$$\vec{\nabla}^2 f=\frac{\partial^2 f}{\partial r^2}+\hat{r}\frac{1}{r}\frac{\partial f}{\partial r}\frac{\partial\hat{r}}{\partial r}+\hat{r}\frac{1}{r}\frac{\partial^2 f}{\partial r\partial\theta}\hat{\theta}+\hat{r}\left(-\frac{1}{r^3}\right)\frac{\partial f}{\partial\theta}\hat{\theta}+\hat{r}\frac{1}{r}\frac{\partial f}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}+\hat{r}\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2 f}{\partial r\partial\varphi}+\hat{r}\left(-\frac{\hat{\varphi}}{r^3\sin\theta}\right)\frac{\partial f}{\partial\varphi}+\hat{r}\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\frac{\partial\hat{\varphi}}{\partial r}+$$

$$+\frac{\hat{\theta}}{r}\frac{\partial^2 f}{\partial\theta\partial r}\cdot\hat{r}+\frac{\hat{\theta}}{r}\frac{\partial f}{\partial r}\frac{\partial\hat{r}}{\partial\theta}+\frac{\hat{\theta}}{r^2}\frac{\partial^2 f}{\partial\theta^2}\cdot\hat{\theta}+\frac{\hat{\theta}}{r^2}\frac{\partial f}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\theta}+\frac{\hat{\theta}}{r^2\sin\theta}\frac{\partial^2 f}{\partial\theta\partial\varphi}\cdot\hat{\varphi}-\frac{\hat{\theta}\cdot\cos\theta}{r^2\sin^2\theta}\frac{\partial f}{\partial\varphi}\cdot\hat{\varphi}+\frac{\hat{\theta}}{r^2\sin\theta}\frac{\partial f}{\partial\varphi}\frac{\partial\hat{\varphi}}{\partial\theta}+$$

$$+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2 f}{\partial\varphi\partial r}\cdot\hat{r}+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial f}{\partial r}\frac{\partial\hat{r}}{\partial\varphi}+\frac{\hat{\varphi}}{r^2\sin\theta}\frac{\partial^2 f}{\partial\varphi\partial\theta}\cdot\hat{\theta}+\frac{\hat{\varphi}}{r^2\sin\theta}\frac{\partial f}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\varphi}+\frac{\hat{\varphi}}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\cdot\hat{\varphi}+\frac{\hat{\varphi}}{r^2\sin^2\theta}\frac{\partial f}{\partial\varphi}\frac{\partial\hat{\varphi}}{\partial\varphi}\xrightarrow[(19)]{(17),(18)}$$

$$\vec{\nabla}^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{\hat{\theta}}{r}\frac{\partial f}{\partial r}\cdot\hat{\theta}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\hat{\theta}}{r^2}\frac{\partial f}{\partial\theta}(-\hat{r})+\frac{\hat{\varphi}}{r}\frac{\partial f}{\partial r}\cdot\hat{\varphi}+$$

$$+\frac{\hat{\varphi}\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta}\cdot\hat{\varphi}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\hat{\varphi}}{r^2\sin^2\theta}\frac{\partial f}{\partial\varphi}\left(-\sin\theta\cdot\hat{r}-\cos\theta\cdot\hat{\theta}=\right)\Leftrightarrow$$

$$\vec{\nabla}^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\Leftrightarrow$$

$$\boxed{\vec{\nabla}^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}}\qquad(22)$$