ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATIONS 1973 - 1982

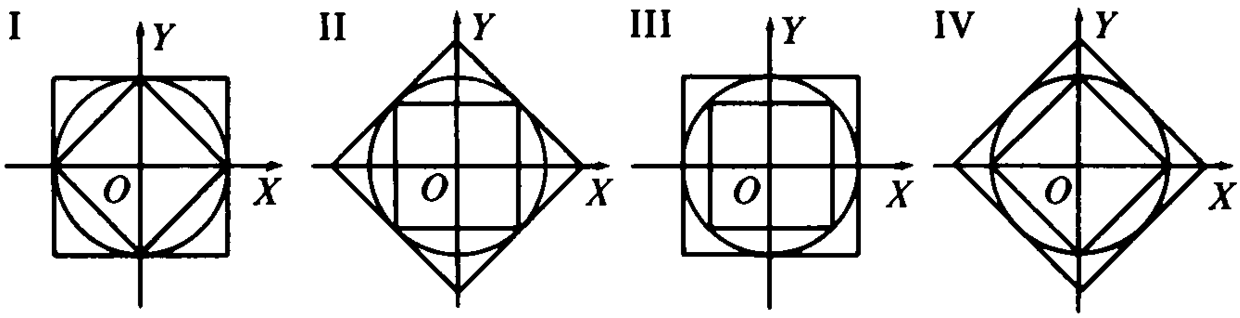
SELECTION TESTS

1973

6. If 554 is the base b representation of the square of the number whose base b representation is 24, then b, when written in base 10, equals

(A) 6 (B) 8 (C) 12 (D) 14 (E) 16

11. A circle with a circumscribed and an inscribed square centered at the origin O of a rectangular coordinate system with positive x and y axes OX and OY is shown in each figure I to IV below.



The inequalities

$$|x| + |y| \leqslant \sqrt{2(x^2 + y^2)} \leqslant 2\,\text{Max}(|x|, |y|)$$

are represented geometrically[†] by the figure numbered

(A) I (B) II (C) III (D) IV (E) none of these

13. The fraction $\dfrac{2(\sqrt{2}+\sqrt{6})}{3\sqrt{2+\sqrt{3}}}$ is equal to

(A) $\dfrac{2\sqrt{2}}{3}$ (B) 1 (C) $\dfrac{2\sqrt{3}}{3}$ (D) $\dfrac{4}{3}$ (E) $\dfrac{16}{9}$

16. If the sum of all the angles except one of a convex polygon is 2190°, then the number of sides of the polygon must be

(A) 13 (B) 15 (C) 17 (D) 19 (E) 21

1974

10. What is the smallest integral value of k such that

$$2x(kx-4)-x^2+6=0$$

has no real roots?

(A) -1 (B) 2 (C) 3 (D) 4 (E) 5

11. If (a, b) and (c, d) are two points on the line whose equation is $y = mx + k$, then the distance between (a, b) and (c, d), in terms of a, c and m, is

(A) $|a-c|\sqrt{1+m^2}$ (B) $|a+c|\sqrt{1+m^2}$ (C) $\dfrac{|a-c|}{\sqrt{1+m^2}}$

(D) $|a-c|(1+m^2)$ (E) $|a-c|\,|m|$

12. If $g(x) = 1 - x^2$ and $f(g(x)) = \dfrac{1-x^2}{x^2}$ when $x \neq 0$, then $f(1/2)$ equals

(A) $3/4$ (B) 1 (C) 3 (D) $\sqrt{2}/2$ (E) $\sqrt{2}$

15. If $x < -2$ then $|1 - |1 + x||$ equals

(A) $2 + x$ (B) $-2 - x$ (C) x (D) $-x$ (E) -2

18. If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

(A) pq (B) $\frac{3p + q}{5}$ (C) $\frac{1 + 3pq}{p + q}$ (D) $\frac{3pq}{1 + 3pq}$
(E) $p^2 + q^2$

1975

10. The sum of the digits in base ten of $(10^{4n^2+8} + 1)^2$, where n is a positive integer, is

(A) 4 (B) $4n$ (C) $2 + 2n$ (D) $4n^2$ (E) $n^2 + n + 2$

12. If $a \neq b$, $a^3 - b^3 = 19x^3$ and $a - b = x$, which of the following conclusions is correct?

(A) $a = 3x$ (B) $a = 3x$ or $a = -2x$
(C) $a = -3x$ or $a = 2x$ (D) $a = 3x$ or $a = 2x$ (E) $a = 2x$

19. Which positive numbers x satisfy the equation $(\log_3 x)(\log_x 5) = \log_3 5$?

(A) 3 and 5 only (B) 3, 5 and 15 only
(C) only numbers of the form $5^n \cdot 3^m$, where n and m are positive integers
(D) all positive $x \neq 1$ (E) none of these

21. Suppose $f(x)$ is defined for all real numbers x; $f(x) > 0$ for all x; and $f(a)f(b) = f(a + b)$ for all a and b. Which of the following statements are true?

I. $f(0) = 1$

II. $f(-a) = 1/f(a)$ for all a

III. $f(a) = \sqrt[3]{f(3a)}$ for all a

IV. $f(b) > f(a)$ if $b > a$

(A) III and IV only (B) I, III and IV only
(C) I, II and IV only (D) I, II and III only (E) All are true.

2. For how many real numbers x is $\sqrt{-(x + 1)^2}$ a real number?

(A) none (B) one (C) two
(D) a finite number greater than two (E) infinitely many

10. If m, n, p and q are real numbers and $f(x) = mx + n$ and $g(x) = px + q$, then the equation $f(g(x)) = g(f(x))$ has a solution

(A) for all choices of m, n, p and q
(B) if and only if $m = p$ and $n = q$
(C) if and only if $mq - np = 0$
(D) if and only if $n(1 - p) - q(1 - m) = 0$
(E) if and only if $(1 - n)(1 - p) - (1 - q)(1 - m) = 0$

19. A polynomial $p(x)$ has remainder three when divided by $x-1$ and remainder five when divided by $x-3$. The remainder when $p(x)$ is divided by $(x-1)(x-3)$ is

(A) $x-2$ (B) $x+2$ (C) 2 (D) 8 (E) 15

20. Let a, b and x be positive real numbers distinct from one. Then

$$4(\log_a x)^2 + 3(\log_b x)^2 = 8(\log_a x)(\log_b x)$$

(A) for all values of a, b and x (B) if and only if $a = b^2$
(C) if and only if $b = a^2$ (D) if and only if $x = ab$
(E) none of these

27. If

$$N = \frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}},$$

then N equals

(A) 1 (B) $2\sqrt{2}-1$ (C) $\frac{\sqrt{5}}{2}$ (D) $\sqrt{\frac{5}{2}}$

(E) none of these

28. Lines $L_1, L_2 \dots, L_{100}$ are distinct. All lines L_{4n}, n a positive integer, are parallel to each other. All lines L_{4n-3}, n a positive integer, pass through a given point A. The maximum number of points of intersection of pairs of lines from the complete set $\{L_1, L_2, \dots, L_{100}\}$ is

(A) 4350 (B) 4351 (C) 4900 (D) 4901 (E) 9851

6. If x, y and $2x + \frac{y}{2}$ are not zero, then

$$\left(2x + \frac{y}{2}\right)^{-1}\left[(2x)^{-1} + \left(\frac{y}{2}\right)^{-1}\right]$$

equals

(A) 1 (B) xy^{-1} (C) $x^{-1}y$ (D) $(xy)^{-1}$
(E) none of these

7. If $t = \dfrac{1}{1 - \sqrt[4]{2}}$, then t equals

(A) $(1 - \sqrt[4]{2})(2 - \sqrt{2})$ (B) $(1 - \sqrt[4]{2})(1 + \sqrt{2})$
(C) $(1 + \sqrt[4]{2})(1 - \sqrt{2})$ (D) $(1 + \sqrt[4]{2})(1 + \sqrt{2})$
(E) $-(1 + \sqrt[4]{2})(1 + \sqrt{2})$

8. For every triple (a, b, c) of non-zero real numbers, form the number

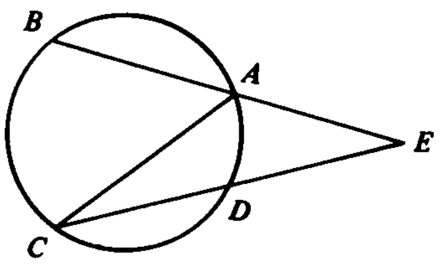
$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}.$$

The set of all numbers formed is

(A) $\{0\}$ (B) $\{-4, 0, 4\}$ (C) $\{-4, -2, 0, 2, 4\}$
(D) $\{-4, -2, 2, 4\}$ (E) none of these

9. In the adjoining figure $\measuredangle E = 40°$ and arc AB, arc BC and arc CD all have equal length. Find the measure of $\measuredangle ACD$.

(A) 10° (B) 15°
(C) 20° (D) $\left(\frac{45}{2}\right)^\circ$
(E) 30°



10. If $(3x - 1)^7 = a_7x^7 + a_6x^6 + \cdots + a_0$, then $a_7 + a_6 + \cdots + a_0$ equals

(A) 0 (B) 1 (C) 64 (D) -64 (E) 128

21. For how many values of the coefficient a do the equations

$$x^2 + ax + 1 = 0,$$
$$x^2 - x - a = 0$$

have a common real solution?

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many

22. If $f(x)$ is a real valued function of the real variable x, and $f(x)$ is not identically zero, and for all a and b

$$f(a+b) + f(a-b) = 2f(a) + 2f(b),$$

then for all x and y

(A) $f(0) = 1$ (B) $f(-x) = -f(x)$ (C) $f(-x) = f(x)$
(D) $f(x+y) = f(x) + f(y)$ (E) there is a positive number T such that $f(x+T) = f(x)$

23. If the solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$, then

(A) $p = m^3 + 3mn$ (B) $p = m^3 - 3mn$ (C) $p + q = m^3$
(D) $\left(\frac{m}{n}\right)^3 = \frac{p}{q}$ (E) none of these

24. Find the sum

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots + \frac{1}{255 \cdot 257}.$$

(A) $\frac{127}{255}$ (B) $\frac{128}{255}$ (C) $\frac{1}{2}$ (D) $\frac{128}{257}$ (E) $\frac{129}{257}$

28. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?

(A) 6 (B) $5 - x$ (C) $4 - x + x^2$ (D) $3 - x + x^2 - x^3$

(E) $2 - x + x^2 - x^3 + x^4$

1978

9. If $x < 0$, then $|x - \sqrt{(x-1)^2}\,|$ equals

(A) 1 (B) $1 - 2x$ (C) $-2x - 1$ (D) $1 + 2x$ (E) $2x - 1$

11. If r is positive and the line whose equation is $x + y = r$ is tangent to the circle whose equation is $x^2 + y^2 = r$, then r equals

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\sqrt{2}$ (E) $2\sqrt{2}$

13. If a, b, c, and d are non-zero numbers such that c and d are the solutions of $x^2 + ax + b = 0$ and a and b are the solutions of $x^2 + cx + d = 0$, then $a + b + c + d$ equals

(A) 0 (B) -2 (C) 2 (D) 4 (E) $(-1 + \sqrt{5})/2$

18. What is the smallest positive integer n such that $\sqrt{n} - \sqrt{n-1} < .01$?

(A) 2499 (B) 2500 (C) 2501 (D) 10,000
(E) There is no such integer.

20. If a, b, c are non-zero real numbers such that

$$\frac{a+b-c}{c} = \frac{a-b+c}{b} = \frac{-a+b+c}{a},$$

and

$$x = \frac{(a+b)(b+c)(c+a)}{abc},$$

and $x < 0$, then x equals

(A) -1 (B) -2 (C) -4 (D) -6 (E) -8

21. For all positive numbers x distinct from 1,

$$\frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x}$$

equals

(A) $\frac{1}{\log_{60} x}$ (B) $\frac{1}{\log_x 60}$ (C) $\frac{1}{(\log_3 x)(\log_4 x)(\log_5 x)}$

(D) $\frac{12}{(\log_3 x) + (\log_4 x) + (\log_5 x)}$

(E) $\frac{\log_2 x}{(\log_3 x)(\log_5 x)} + \frac{\log_3 x}{(\log_2 x)(\log_5 x)} + \frac{\log_5 x}{(\log_2 x)(\log_3 x)}$

1979

22. Find the number of pairs (m, n) of integers which satisfy the equation $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1$.

(A) 0 (B) 1 (C) 3 (D) 9 (E) infinitely many

25. If $q_1(x)$ and r_1 are the quotient and remainder, respectively, when the polynomial x^8 is divided by $x + \frac{1}{2}$, and if $q_2(x)$ and r_2 are the quotient and remainder, respectively, when $q_1(x)$ is divided by $x + \frac{1}{2}$, then r_2 equals

(A) $\frac{1}{256}$ (B) $-\frac{1}{16}$ (C) 1 (D) -16 (E) 256

26. The function f satisfies the functional equation

$$f(x) + f(y) = f(x + y) - xy - 1$$

for every pair x, y of real numbers. If $f(1) = 1$, then the number of integers $n \neq 1$ for which $f(n) = n$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinite

1980

8. How many pairs (a, b) of non-zero real numbers satisfy the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}?$$

(A) none (B) 1 (C) 2 (D) one pair for each $b \neq 0$
(E) two pairs for each $b \neq 0$

14. If the function f defined by

$$f(x) = \frac{cx}{2x+3}, \quad x \neq -\frac{3}{2}, \quad c \text{ a constant,}$$

satisfies $f(f(x)) = x$ for all real numbers x except $-\frac{3}{2}$, then c is

(A) -3 (B) $-\frac{3}{2}$ (C) $\frac{3}{2}$ (D) 3

(E) not uniquely determined by the given information

27. The sum $\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}$ equals

(A) $\frac{3}{2}$ (B) $\frac{\sqrt[3]{65}}{4}$ (C) $\frac{1+\sqrt[6]{13}}{2}$ (D) $\sqrt[3]{2}$

(E) none of these

28. The polynomial $x^{2n}+1+(x+1)^{2n}$ is not divisible by x^2+x+1 if n equals

(A) 17 (B) 20 (C) 21 (D) 64 (E) 65

1981

6. If $\frac{x}{x-1}=\frac{y^2+2y-1}{y^2+2y-2}$, then x equals

(A) y^2+2y-1 (B) y^2+2y-2 (C) y^2+2y+2
(D) y^2+2y+1 (E) $-y^2-2y+1$

10. The lines L and K are symmetric to each other with respect to the line $y=x$. If the equation of line L is $y=ax+b$ with $a\neq 0$ and $b\neq 0$, then the equation of K is $y=$

(A) $\frac{1}{a}x+b$ (B) $-\frac{1}{a}x+b$ (C) $-\frac{1}{a}x-\frac{b}{a}$

(D) $\frac{1}{a}x+\frac{b}{a}$ (E) $\frac{1}{a}x-\frac{b}{a}$

15. If $b>1$, $x>0$ and $(2x)^{\log_b 2}-(3x)^{\log_b 3}=0$, then x is

(A) $\frac{1}{216}$ (B) $\frac{1}{6}$ (C) 1 (D) 6

(E) not uniquely determined

17. The function f is not defined for $x = 0$, but, for all non-zero real numbers x,

$f(x) + 2f\left(\frac{1}{x}\right) = 3x$. The equation $f(x) = f(-x)$ is satisfied by

(A) exactly one real number
(B) exactly two real numbers
(C) no real numbers
(D) infinitely many, but not all, non-zero real numbers
(E) all non-zero real numbers

18. The number of real solutions to the equation

$$\frac{x}{100} = \sin x$$

is

(A) 61 (B) 62 (C) 63 (D) 64 (E) 65

1982

1. When the polynomial $x^3 - 2$ is divided by the polynomial $x^2 - 2$, the remainder is

(A) 2 (B) -2 (C) $-2x - 2$ (D) $2x + 2$ (E) $2x - 2$

6. The sum of all but one of the interior angles of a convex polygon equals $2570°$. The remaining angle is

(A) $90°$ (B) $105°$ (C) $120°$ (D) $130°$ (E) $144°$

12. Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then $f(7)$ equals

(A) -17 (B) -7 (C) 14 (D) 21
(E) not uniquely determined

13. If $a > 1, b > 1$ and $p = \dfrac{\log_b(\log_b a)}{\log_b a}$, then a^p equals

(A) 1 (B) b (C) $\log_a b$ (D) $\log_b a$ (E) $a^{\log_b a}$

17. How many real numbers x satisfy the equation

$$3^{2x+2} - 3^{x+3} - 3^x + 3 = 0\,?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

SOLUTIONS

16. (B) Let n denote the number of sides of the given convex polygon and x the number of degrees in the excepted angle. Then $180(n-2) = 2190 + x$, so that

$$n - 2 = \frac{2190}{180} + \frac{x}{180}.$$

Since the polygon is convex, $0 < x < 180$; it follows that

$$\frac{2190}{180} < n - 2 < \frac{2190}{180} + 1,$$

i.e. $12\frac{1}{6} < n - 2 < 13\frac{1}{6}$. Since n is an integer, this forces $n - 2 = 13$, so $n = 15$. (Incidentally, $(13)(180) = 2340 = 2190 + 150$, so the excepted angle has measure $150°$.)