1605. *Proposed by M.S. Klamkin and Andy Liu, University of Alberta.*

ADB and AEC are isosceles right triangles, right-angled at D and E respectively, described outside ΔABC. F is the midpoint of BC. Prove that DFE is an isosceles right-angled triangle.

7. [1989: 68] *24th Spanish Olympiad.*
Solve the following system of equations in the set of complex numbers:

$$|z_1| = |z_2| = |z_3| = 1,$$
$$z_1 + z_2 + z_3 = 1,$$
$$z_1 z_2 z_3 = 1.$$

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let $\overline{z}$ denote the complex conjugate of z. We have $\overline{z}_i = 1/z_i$ for $i = 1, 2, 3$. It follows that

$$z_1 z_2 + z_2 z_3 + z_3 z_1 = \frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2} = \overline{z}_3 + \overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2 + z_3} = 1.$$

Consider the cubic polynomial

$$(x - z_1)(x - z_2)(x - z_3) = x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1).$$

Since $1, \pm i$ are the roots we have that z_1, z_2, z_3 are equal to $1, i, -i$ in some order.

11. The domain of function f is $[0,1]$, and for any $x_1 \neq x_2$

$$|f(x_1) - f(x_2)| < |x_1 - x_2|.$$

Moreover, $f(0) = f(1) = 0$. Prove that for any x_1, x_2 in $[0,1]$,

$$|f(x_1) - f(x_2)| < \frac{1}{2}.$$

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh, and by Michael Selby, University of Windsor.

First $|f(x) - f(0)| \leq |x - 0|$, i.e. $|f(x)| \leq x$ and the inequality is strict for $x \neq 0$. Also $|f(x) - f(1)| \leq |x - 1|$, i.e. $|f(x)| \leq 1 - x$ with strict inequality for $x \neq 1$. Therefore

$$|f(x)| \leq \min(x, 1 - x),$$

and the inequality is strict unless $x = 0$ or $x = 1$. Let $x_1, x_2 \in [0,1]$. If $|x_1 - x_2| \leq 1/2$, then

$$|f(x_1) - f(x_2)| \overset{*}{\leq} |x_1 - x_2| \leq 1/2.$$

This gives $|f(x_1) - f(x_2)| < 1/2$ since * is strict unless $x_1 = x_2$ and this case is trivial. So suppose $|x_1 - x_2| > 1/2$. Without loss of generality suppose that $x_1 \in (1/2, 1]$ and $x_2 \in [0, 1/2)$. Then

$$|f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| \leq 1 - x_1 + x_2 = 1 - (x_1 - x_2) < \frac{1}{2}.$$

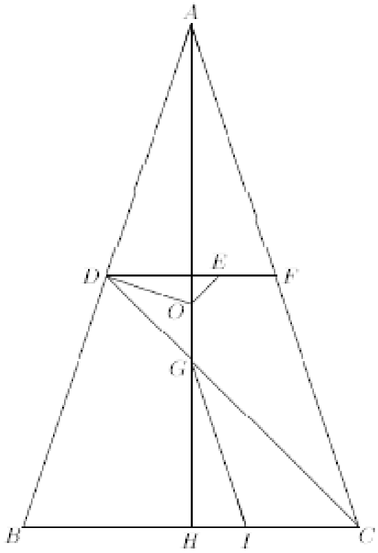
Therefore, in all cases $|f(x_1) - f(x_2)| < 1/2$ for all $x_1 \neq x_2$.

1. [1983: 108] *1983 British Mathematical Olympiad.*

In the triangle ABC with circumcentre O , $AB = AC$, D is the midpoint of AB, and E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

Solution by Jie Lou, student, Halifax West High School.

Join the lines DE, DO, and AO, and let F be the intersection of DE and AC, G the intersection of AO and CD, and H the intersection of AO and BC. Find the point I on BC such that $HI = \frac{1}{3}HC$. Since ΔABC is isosceles and O is the circumcentre, AO is the central line of BC. Since D is the midpoint of AB, G is the centroid of the triangle, and $GH = \frac{1}{3}AH$. Thus $GI \| AC$. Therefore $\angle HGI = \angle HAC = \angle DAO$. Since O is the circumcentre and D is the midpoint of AB, OD is perpendicular to AB. Also, we have $\angle GHI = 90^\circ$. Then $\Delta GHI \sim \Delta ADO$. From this we have $GH/AD = HI/DO$. Now, since $DE = \frac{2}{3}DF = \frac{2}{3}CH = 2HI$ and $AG = 2GH$, we have



$$\frac{AG}{AD} = \frac{DE}{DO}.$$

Obviously, DE is perpendicular to AH, so that $\angle ODE = 90^\circ \Leftrightarrow \angle ADE = \angle DAG$. From this, $\Delta ADG \sim \Delta DOE$. Since the angle between AD and DO is 90°, the angle between DG and EO must be 90°, too. Thus OE is perpendicular to CD.

1639. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$ABCD$ is a convex cyclic quadrilateral. Prove that

$$(AB + CD)^2 + (AD + BC)^2 \geq (AC + BD)^2.$$

For the remainder of this column, we turn to problems given in the June 1989 number of the Corner. We give solutions to all but numbers 3 and 6 of the *3rd Ibero-American Olympiad* [1989: 163–164].

1. The angles of a triangle are in arithmetical progression. The altitudes of the triangle are also in arithmetical progression. Show that the triangle is equilateral.

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.

Let A, B, C be the angles of the given triangle and let h_a, h_b, h_c be the corresponding altitudes. Without loss of generality, we may assume $A \le B \le C$. Since the angles are in arithmetic progression $A + C = 2B$, and since $A + B + C = 180^\circ$, $B = 60^\circ$. Now also $h_c \le h_b \le h_a$ and $a \le b \le c$ where a, b, c are the side lengths opposite A, B, C, respectively.

From the law of cosines $b^2 = a^2 + c^2 - 2ac\cos 60^\circ = a^2 + c^2 - ac$. Now $2h_b = h_a + h_c$ implies that $4F/b = 2F/a + 2F/c$, where F is the area of triangle ABC, so

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac}\ , \qquad \text{or} \qquad b = \frac{2ac}{a+c}.$$

Since $b^2 = a^2 + c^2 - ac$, $4a^2c^2 = (a+c)^2(a^2 + c^2 - ac)$. From this we get

$$\begin{aligned} 0 &= (a+c)^2(a^2 + c^2 - ac) - 4a^2c^2 \\ &= [(a-c)^2 + 4ac][(a-c)^2 + ac] - 4a^2c^2 \\ &= (a-c)^4 + 5ac(a-c)^2 \\ &= (a-c)^2[(a-c)^2 + 5ac], \end{aligned}$$

and so $a = c$. This gives $a = b = c$ since $a \le b \le c$, and the given triangle is equilateral.

1818. [1993: 50] *Proposed by Ed Barbeau, University of Toronto.*
Prove that, for $0 \leq x \leq 1$ and a positive integer k,

$$(1+x)^k[x+(1-x)^{k+1}] \geq 1.$$

I. *Solution by Panos E. Tsaoussoglou, Athens, Greece.*
The proof is by induction on k.
Let $k=1$; then

$$(1+x)[x+(1-x)^2] = (1+x)(1-x+x^2) = 1+x^3 \geq 1,$$

and thus the inequality is true for $k=1$.

Assume that the inequality holds for some $k=n \geq 1$, i.e., $(1+x)^n[x+(1-x)^{n+1}] \geq 1$. It is sufficient to prove that

$$(1+x)^{n+1}[x+(1-x)^{n+2}] - (1+x)^n[x+(1-x)^{n+1}] \geq 0,$$

or equivalently that

$$x(1+x)^{n+1} + (1+x)^{n+1}(1-x)^{n+2} - x(1+x)^n + (1+x)^n(1-x)^{n+1} \geq 0.$$

However, the left hand side reduces to

$$\begin{aligned} &x(1+x)^n(1+x-1) + (1+x)^n(1-x)^{n+1}(1-x^2-1) \\ &\qquad = x^2(1+x)^n - x^2(1+x)^n(1-x)^{n+1} = x^2(1+x)^n[1-(1-x)^{n+1}]. \end{aligned}$$

Thus we have to show that

$$x^2(1+x)^n[1-(1-x)^{n+1}] \geq 0.$$

But this inequality is true since $1 \geq x \geq 0$ and $1 \geq (1-x)^{n+1}$. Therefore, the given inequality is true for all k.

II. *Solution by Chris Wildhagen, Rotterdam, The Netherlands.*
We have

$$\begin{aligned} (1+x)^k[x+(1-x)^{k+1}] &= x(1+x)^k + (1-x)(1-x^2)^k \\ &\geq [x(1+x)+(1-x)(1-x^2)]^k \qquad (1) \\ &= (1+x^3)^k \geq 1, \end{aligned}$$

where for (1) the convexity of the function $t \mapsto t^k$ $(k \geq 1)$ on the interval $[0,\infty)$ is used. [*Editor's note*: (1) uses Jensen's inequality; we could also use the fact that the (weighted) kth power mean for $k > 1$ is greater than the arithmetic mean.]

Now we turn to solutions to problems of the 22nd *Austrian Mathematical Olympiad 2nd Round* [1993: 101]. (Next month we will give the solutions received for the final round.)

1. Let a, b be rational numbers such that $\sqrt[3]{a}+\sqrt[3]{b}$ is a rational number $c \neq 0$. Show that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

Solutions by Seung-Jin Bang, Albany, California; by Joel Brenner, Palo Alto, California; by Geoffrey A. Kandall, Hamden, Connecticut; by Waldemar Pompe, student, University of Warsaw, Poland; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Kandall's solution which was similar to several others.

We have $c^3 = a+b+3\sqrt[3]{a}\,\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})$, hence we have $\sqrt[3]{a}\,\sqrt[3]{b} = (c^3-a-b)/3c \in \mathbf{Q}$. Let $k = (\sqrt[3]{a})^2 + \sqrt[3]{a}\,\sqrt[3]{b} + (\sqrt[3]{b})^2$. Note that $k = c^2 - \sqrt[3]{a}\,\sqrt[3]{b} \in \mathbf{Q}$. We have $a-b = (\sqrt[3]{a}-\sqrt[3]{b})k$. If $k \neq 0$ then $\sqrt[3]{a}-\sqrt[3]{b} = (a-b)/k \in \mathbf{Q}$. If $k=0$ then $a=b$. So, in either case $\sqrt[3]{a}-\sqrt[3]{b} \in \mathbf{Q}$.

Finally $\sqrt[3]{a} = \frac{1}{2}((\sqrt[3]{a}+\sqrt[3]{b})+(\sqrt[3]{a}-\sqrt[3]{b})) \in \mathbf{Q}$, and $\sqrt[3]{b} = c - \sqrt[3]{a} \in \mathbf{Q}$.

2. Determine all real solutions of the equation

$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14}=0.$$

3. Find all natural numbers n which satisfy equalities

$$S(n) = S(2n) = S(3n) = \cdots = S(n^2)$$

if $S(x)$ denotes the sum of digits of the number x (in decimal).

Solutions by Himadri Choudhury, student, Hunter High School, New York; and by Bob Prielipp, University of Wisconsin-Oshkosh. We give Prielipp's solution.

The natural numbers n which satisfy our equalities are 1 and $10^m - 1$ for $m = 1, 2, \ldots$. To prove this assertion, we begin with some simple, but useful, facts about digital sums. Let $L(x)$ denote the number of large digits (digits greater than or equal to 5) in the number x and let $C(x \oplus y)$ denote the number of carries when x and y are added using the normal algorithm of addition. Then

$$\begin{aligned} S(2n) &= 2S(n) - 9L(n) \\ S(m+n) &= S(m) + S(n) - 9C(m \oplus n) \\ S((10^m - 1) - n) &= 9m - S(n). \end{aligned}$$