57-th Romanian Mathematical Olympiad 2006

Final Round

Iaşi, April 17, 2006

7-th Form

1. Consider points M and N on respective sides AB and BC of a triangle ABC such that $2CN/BC = AM/AB$. Let P be a point on line AC. Prove that lines MN and NP are perpendicular if and only if PN bisects $\angle MPC$.

2. A square of side n is divided into n^2 unit squares each of which is colored red, yellow or green. Find the smallest value of n such that, for any coloring, there exist a row and a column having at least three equally colored unit squares.

3. In an acute triangle ABC, the angle at C equals $45°$. Points A_1 and B_1 are the feet of the altitudes from A and B respectively, and H is the orthocenter. Points D and E are taken on segments AA_1 and BC respectively such that $A_1D = A_1E = A_1B_1$. Show that

   (a) $A_1B_1 = \sqrt{\dfrac{A_1B^2 + A_1C^2}{2}}$;

   (b) $CH = DE$.

4. Let A be a set of at least two positive integers. Suppose that for each $a, b \in A$ with $a > b$ we have $\frac{[a,b]}{a-b} \in A$. Show that set A has exactly two elements.

8-th Form

1. In a hexagonal prisma, five of the six lateral faces are cyclic quadrilateral. Prove that the remaining lateral face is also a cyclic quadrilateral.

2. Let n be a positive integer. Show that there exist an integer $k \geq 2$ and numbers $a_1, a_2, \ldots, a_k \in \{-1, 1\}$ such that

$$n = \sum_{1 \leq i < j \leq k} a_i a_j.$$

3. Let $ABCDA'B'C'D'$ be a cube and P be a variable point on edge AB. A plane through P perpendicular to AB intersects line AC' at point Q. Denote by M and N the midpoints of $A'P$ and BQ, respectively.

   (a) Prove that MN and BC' are perpendicular if and only if P is the midpoint of AB.

(b) Find the smallest measure of the angle between MN and BC'.

4. Prove that if $\frac{1}{2} \le a, b, c \le 1$, then

$$2 \le \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \le 3.$$

9-th Form

1. Find the maximum value of the expression $(x^3+1)(y^3+1)$ if x and y are real numbers with $x+y=1$.

2. The isosceles triangles ABC and DBC have the common base BC and $\angle ABD = 90°$. Let M be the midpoint of BC. Points E, F, P are such that E, P and C are interior to the segments AB, MC and AF respectively, and $\angle BDE = \angle ADP = \angle CDF$. Show that P is the midpoint of EF and $DP \perp EF$.

3. A quadrilateral $ABCD$ is inscribed in a circle of radius r so that there exists a point P on side CD satisfying $CB = BP = PA = AB$.

   (a) Show that such points A, B, C, D, P indeed exist.

   (b) Prove that $PD = r$.

4. A tennis tournament with $2n$ participants ($n \ge 5$) extends over 4 days. Each day, every participant plays exactly one match (but one pair may meet more than once). After the tournament, it turns out that there is a single winner and three players sharing the second place, and that there is no player who lost all four matches. How many players did win exactly one and two matches, respectively?

10-th Form

1. Let M be an n-element subset and $\mathcal{P}(M)$ be he set of all its subsets (including $\emptyset$ and M). Determine all functions $f : \mathcal{P}(M) \to \{0, 1, \dots, n\}$ with the following properties:

   (a) $f(A) \ne 0$ for $A \ne \emptyset$;

   (b) $f(A \cup B) = f(A \cap B) + f(A \Delta B)$ for all $A, B \in \mathcal{P}(M)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

2. Prove that for all $a, b \in (0, \frac{\pi}{4})$ and $n \in \mathbb{N}$

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \ge \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}.$$

3. Show that the sequence given by $a_n = \left[n\sqrt{2}\right] + \left[n\sqrt{3}\right]$, $n = 0, 1, \dots$ contains infinitely many even numbers and infinitely many odd numbers.

4. Given an integer $n \ge 2$, determine n pairwise disjoint sets A_i, $1 \le i \le n$ with the following properties:

   (a) For every circle $\mathcal{C}$ in the plane and for all i, $A_i \cap Int(C) \ne \emptyset$;

   (b) For every line d in the plane and for all i, the projection of A_i onto d is all of d.

11-th Form

1. Let A be a complex $n \times n$ matrix and let A^* be its adjoint matrix. Prove that if there exists an integer $m \geq 1$ such that $(A^*)^m = 0$, then $(A^*)^2 = 0$.

2. We say that a matrix $B \in \mathcal{M}_n(\mathbb{C})$ is a *pseudoinverse* of $A \in \mathcal{M}_n(\mathbb{C})$ if $A = ABA$ and $B = BAB$.

   (a) Show that every square matrix has a pseudoinverse.

   (b) For which matrices is there a unique pseudoinverse?

3. Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be distinct points in the plane. Show that there exists a point P such that
$$PA_1 + PA_2 + \cdots + PA_n = PB_1 + PB_2 + \cdots + PB_n.$$

4. Consider a function $f : [0, \infty) \to \mathbb{R}$ with the property that for each $x > 0$, the sequence $f(nx)$, $n = 0, 1, 2, \ldots$ is strictly increasing.

   (a) If f is continuous on $[0, 1]$, is f necessarily strictly increasing?

   (b) The same question if f is continuous on $\mathbb{Q}^+$.

12-th Form

1. Let K be a finite field. Prove that the following statements are equivalent:

   (a) $1 + 1 = 0$;

   (b) For each $f \in K[x]$ with $\deg f \geq 1$, $f(x^2)$ is reducible.

2. Prove that
$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}dx\right) = \int_0^1 f(x)dx,$$
where $f(x) = \dfrac{\arctan x}{x}$ for $0 < x \leq 1$ and $f(0) = 1$.

3. Let G be an n-element group ($n \geq 2$) and let p be a prime factor of n. Prove that if G has a unique subgroup H of p elements, then H is contained in the center of G. (The center of G is $Z(G) = \{a \in G \mid ax = xa, \forall x \in G\}$.)

4. A function $f : [0, 1] \to \mathbb{R}$ satisfies $\displaystyle\int_0^1 f(x)dx = 0$. Prove that there exists $c \in (0, 1)$ such that
$$\int_0^c xf(x)dx = 0.$$