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THE MEANING OF CONICS: HISTORICAL AND DIDACTICAL DIMENSIONS

In this paper, I draw on an analysis of the research project *Mathematical Machines,* which concerns the teaching and learning of geometry in high school (grades 9 to 13). Although the project is actually broader (see Bartolini Bussi, 1993, 1998, 2000, 2001; Bartolini Bussi & Mariotti, 1999; Bartolini Bussi et al., 1999; Bartolini Bussi & Pergola, 1994, 1996), I have chosen the special topic of *conic sections* (or *conics*), which I take to be representative of the whole approach.

My main thesis is that *the present meaning of conics is the result of the complex relationships between the different processes of studying conics during different historical ages*, each of which has left a residue in the names, the problems, the means of representations, the rules of actions, and the systems of control. To investigate the present meaning, we may refer to the historical development of the study of conics by means of *time periods*, each framed in the culture of a different age. Even if from today's standpoint all the conics studied in the different time periods can be identified as the same objects, inside each time period different meanings have been built by geometers to the extent that conics are representative of the development of different conceptualisations of space and geometry over the ages. As a corollary, I claim that *it is not possible to build the meaning of conics through only a one-sided approach*, as, for instance, through the most widespread algebraic definition.

If history is an unavoidable component of the construction of meaning, a didactic problem immediately arises: *How is it possible to introduce students to the historical problematic without undue oversimplification?*

An exemplary teaching experiment will be described to show how the problem of *epistemological complexity* (as meant by Hanna & Jahnke, 1994), on the one hand, and the problem of *historical contextualisation,* on the other, are coped with by means of a selection of tasks. Finally, a small-group study of a special model for the parabola (the *orthotome,* inherited from the Greek tradition) will be analysed. It concerns how the meaning is constructed by students through the introduction of a conscious anachronism that fosters an intentional recourse to different tools developed in different ages and allows the students to relate different ways of representation to each other.

1. THE MEANING OF CONICS: HISTORICAL DIMENSION

Consider the special topic of *conics* (for centuries called *conic sections* to emphasise the generation by means of a cone). I claim that the most common approach, which is through analytic geometry (i.e., conics as plane loci satisfying equations of the second degree obtained from some metric relations), is not enough. A lot of the meaning of conics is lost: Where do their names come from? Why are they studied together? Why do they have some special importance in geometry? And so on.

The algebraic approach is deepened, if, as happens in university courses, geometry is replaced by linear algebra and more notions are added. Quadratic forms in three variables are considered as special cases of quadratic forms in $n + 1$ variables that define hyperquadrics in n-dimensional complex projective space: Terms like *cone, cylinder, diameter, axis,* and so on are used. Why? Bourbaki (1974) claims, in his historical reconstruction of the genesis of quadratic forms, that in the search for greater and greater "abstraction", it has been considered very suggestive and attractive to preserve the terminology that originated in the study of cases of two and three dimensions from classical geometry and to extend it to the case of n dimensions, to the extent that geometry has been transformed into an universal language for contemporary mathematics. But surely the intention to convey suggestion and attraction can be realised only for people who also know the spatial generation of conics.

Asked the question, What is the meaning of conics? (which is related to the question of the meaning of quadratic forms), one can give many answers. To cite just a few, one can consider conics analytically as curves of second degree, synthetically in three-dimensional space as conic sections, in the plane as loci satisfying some metric conditions, as perceived images of a circle from a variable point of view, and so on. All these interpretations are related to each other, yet they are concerned with different conceptualisation of conics that can be related now by means of the existing body of knowledge.

To explain the above discourse, it is necessary to reconstruct the historical development of conics. Obviously, the figural representations of conics (as signs traced by means of a gesture, linkage, or cut made either in the sand, on paper, in the air, or on the surface of a cone, and so on) are invariant in time and hence not subject to historical changes. What are changed are the way of generating conics, the way of looking at them, and the way of studying them.

These attitudes are actually more general. The history of conics (as well as the history of any mathematical object that dates back to antiquity and is still part of today's mathematical culture) is a metonymy for a more general history, namely, the history of the geometrical conceptualisation of space. As such, it cannot be understood inside mathematics only and requires references to the complex relationships between mathematics and general culture.

In short (for details, see Coolidge, 1945), one may identify four major phases:

1. Greek mathematics, where the early emergence of conic sections is documented.
2. The 17^{th} century.

3. The 18th and 19th centuries.
4. The 20th century.

The two initial phases are of critical importance: The first concerns the birth of conics as geometric objects; the second concerns the emergence of the trends of discussion that characterise the modern treatment of conics. The jump from Greek mathematics to the 17th century is due not to a naive underestimation of the contributions of the Middle Ages and the Renaissance (they actually constitute the ground on which subsequent development is based) but to the fact that in the 17th century one encounters the early results of a complex social phenomenon that radically changed the attitude towards mathematics, the sciences, and technology—that is, the interaction of merchants, scientists, engineers, artists, medical practitioners, humanists, and so on, and its amplification through the increasingly widespread diffusion of ideas by means of the printing press (Otte, 1993). The eruption of new ideas in mathematics is visible also in the approach to conics, where new conceptual tools were being accepted from outside mathematics: for example, from commercial arithmetic (i.e., algebra), from the arts (i.e., the introduction of points at infinity in perspective drawing and the study of anamorphoses in painting), and from technology (i.e., machines for drawing curves).

In the next two centuries, this new attitude was developed and carried to extremes: The complete algebraisation of conics allowed the development of the theory of quadratic forms in connection with problems from arithmetic, analysis, and mechanics; the great projective school allowed the characterisation of conics on the basis of location and intersection rather than of metric properties; the theory of articulated systems, developed in connection with kinematics, on the one hand, and the theoretical treatment of geometric transformations, on the other, allowed the characterisation of algebraic curves (which include conics) as the curves that can be traced by linkages. Later, in the 20th century, the Bourbakist program for the complete algebraisation of conics up to the theory of quadratic forms had a great effect on devisualisation, and only in the last decade of the century did the introduction of computer aids reintroduce a visual dimension into that purely algebraic world.

Within each period of time, different objects were built by geometers. What the objects have in common is the name (and sometimes not even that, as we shall see) and some classical problems. The difference is so deep that mathematicians often feel obliged *to prove* that the new objects are actually the same as in the past.

There is not space in this chapter to offer an account of the different time periods. For the purpose of this paper, I limit myself to reminding the reader that at the beginning of the story, a parabola was obtained by Greek geometers by cutting a right-angled right circular cone with a plane perpendicular to one element of the cone (whence the Greek name *orthotome*), whilst the curves today known as ellipse and hyperbola were obtained by cutting an acute-angled and an obtuse-angled cone (whence the Greek names of *oxytome* and *amblytome*) in the same way (some details on these issues are in Coolidge, 1945; a proof in the case of orthotome is given in the section "Students' construction of meaning" below).

Later, things changed. For instance, Descartes (Bos, 1981) treated the curves in totally different ways according to their role as either a solution of a problem (a product) or a means of finding solutions. A pointwise construction of a conic by ruler and compass is sufficient when it occurs as a solution, but when in a problem it is necessary to find solutions by intersecting conics, a pointwise construction is no longer sufficient. A stronger criterion is required: It is necessary to have a method to trace the curve by means of "some regular motion" that allows one to find *all* the points so that the intersections with other lines are precisely (and not only approximately) constructable. The problem of continuity is solved by referring to motion and time. In the same fashion, further developments added new elements to the meaning of conics to constitute a complex object.

2. THE MEANING OF CONICS: DIDACTIC DIMENSION

The present meaning of conics is the result of an accretion of terms, problems, ways of representations, rules of actions and systems of control that have been inherited from the different time periods. This fact has important consequences on the didactic plane: To construct the meaning of conics in the classroom, it is necessary to reconstruct some elements of their historical development.

This is actually a didactic problem: Is it possible to introduce students to the historical problematic without undue oversimplification? In what way? This section of the paper is devoted to that issue, showing how my colleagues and I designed and implemented a field of experience for students' activity in the classroom in order to implement the historical reconstruction of the meaning of conics. The experiments were carried on in secondary classrooms up to the late 1990s. Later, they were shifted to the university level and to pre-service teacher education.

The context of classroom activity is characterised by the presence of physical models (either static or dynamic), which are not simply shown to students but are objects for students' investigation. So, for instance, students are given a three-dimensional model of a conic section or a plane trammel that draws a conic, and the task is to determine the geometrical properties of the points on the curve. These models are historically contextualised by means of guided reading of historical sources.

I illustrate this process by means of three kinds of data:

a) a scheme of a long-term teaching experiment designed and implemented in the classroom to realise the main motives of the whole activity,
b) the analysis of a task whose goals are consistent with the motives of the whole activity,
c) the analysis of a small-group session up to the product of a collective written text.

2.1 *Motives of the teaching-learning activity for Mathematical Machines*

The teaching-learning activity in the research project *Mathematical Machines* is polimotivated. The term *motives* is used after Leont'ev (1978) to mean the objects of an *activity*, to be distinguished from *goals* (or aims) and *conditions*. The macrostructure of an activity consists of *actions*, each of which is directed to a specific goal; the ways of realising actions in concrete conditions are *operations*. The study of long-term processes concerns the relationships between the levels of teacher's motives, actions, and operations and the effects produced on students' activity.

The motives can be briefly sketched as follows (they are surely mutually intertwined):

1. the conceptualisation of mathematics not as an isolated body of knowledge but as a part of the global cultural development of mankind to be studied in its relationships with other fields of knowledge,
2. the historical contextualisation of accepted rules of behaviour,
3. the multifaceted meaning of conics aside from the more usual plane meaning as loci of points defined by metric relations,
4. the dynamic interpretation of either dynamical or static objects used to guess conjectures and to guide the construction of early proofs, plus the introduction of movements and of the principle of continuity (implicitly) to cope with the problem of "generic" points

The motives can be determined a priori by the analysis of the teacher's programmatic documents (Pergola & Zanoli, 1994, 1995) and can be checked a posteriori by either the definition of school tasks or the quality of interaction in the classroom. Below, I give the scheme of the tasks of a particular teaching experiment and some exemplary excerpts of small-group interaction during the study of the orthotome.

2.2 *Tasks in the laboratory of Mathematical Machines*

Student activity takes place in a special room (the mathematical laboratory), where several physical models (either static or dynamic) are at the students' disposal (two catalogues of the models can be found on line at http://www.mmlab.unimore.it). Large-sized models (built on bases that are more than 60 cm by 60 cm) have been built by the teachers themselves using wood, brass, plexiglas, coloured threads, sinkers, and so on. Models are sometimes used by the teacher to illustrate a concept, but more often they are handled by the students themselves in order to examine them according to some specific task. Concrete handling of models is contextualised by means of the guided reading of some selected and annotated historical sources.

For the special topic under scrutiny, several models are available for either solid or plane study and for either static contemplation or the dynamic generation of conics. We have models from the Greek period (e.g., models from Menaechmus and Apollonius) and models from the 16^{th} to 20^{th} centuries (e.g., models for the mechanical generation and the projective study of conics). However, the models alone are opaque; only the reading of sources and guided manipulation can make the different conceptualisations explicit.

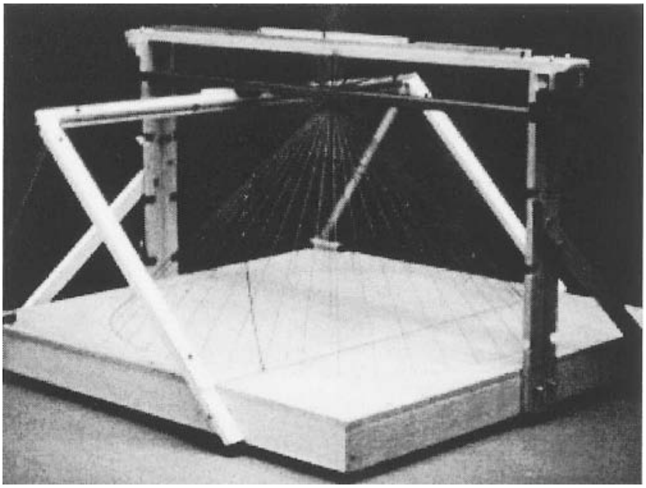

Figure 1: The model of the orthotome studied by students in small-group work

2.3 A teaching experiment

Long-term teaching experiments are designed and implemented in the laboratory as standard part of the curriculum. In some cases, the model substitutes for the need for an explicit proof; in other cases, it generates the need for a proof. In any case, it allows the teacher to introduce historical digressions that contextualise the study in the culture of the corresponding age. The first part of an exemplary teaching experiment is described below (see Tufo, 1995).

2.3.1 The teacher's lesson: A historical introduction to conics in ancient Greece

The teacher motivates the historical introduction with the need to reconstruct a deep geometrical meaning for the algebraic relationships between coordinates (already known by students and applied to a classical derivation of conics from focal properties) that represent conics in the Cartesian plane. He illustrates the conceptual difference between the way of looking at conic sections in ancient Greece and at conics in the modern age (since the 16th century). He focuses on three issues:

1. conics as solid curves versus conics as plane curves,
2. conics as existing objects to be studied (plane sections of cones) versus conics as products (representations of laws or drawings by instruments),
3. synthetic versus analytic study of conics

2.3.2 Small-group work: The study of the orthotome model

The whole class is divided into five small groups, each studying a different model. A small group of four students is given the task of studying the model of the *orthotome* and of deriving the "symptom" of the *parabola.* The students are prompted to refer to theorems on right triangles and on similar right triangles and to avoid recourse to coordinates, as coordinates and analytic geometry did not exist in ancient Greece. They have to state the symptom and to prove it, in order to be able to explain it to their schoolmates in a later lesson. Small-group work is carried on with some interventions by the teacher, who walks around the classroom to observe the groups at work. A final written report is requested from the group (see below).

2.3.3 Students' explanation of the orthotome model

Two students in the group present to the whole class the result and the proof arrived at in their small-group work.

2.3.4 The teacher's lesson: Models of the orthotome and the equations of conics

The model of the orthotome is considered again: A system of coordinates is introduced to derive the canonical equation of the parabola

$$2\,k\,x = y^2$$

2.3.5 The rest of the experiment

A leap is made to the 17th century to introduce the study of conics according to de l'Hospital. The teacher introduces this different approach by means of conic-drawing instruments, discussing the changes in attitudes towards geometry. Then he proposes the study of conics by means of three different definitions from de l'Hospital, based on conic-drawing instruments. They are quite different from each other and make it clear than the conceptualisation of ellipse, hyperbola, and parabola as different manifestations of the same geometrical object is quite difficult from a metric perspective.

2.3.6 Students' construction of meaning

In this section, I discuss the small-group work up to the collective essay produced by the students at the end of their study of the orthotome model. The final text was produced collectively after a 2-hour small-group laboratory on the model of the orthotome. In this section, I use a two-column format: In the right column are transcripts of discussions and texts produced by the students; in the left column are the students' original drawings and comments by my colleagues and me.

2.3.7 The quality of small group interaction

The task, assigned verbally by the teacher, was the following (according to the complete transcript of the laboratory; see Tufo, 1995):

Teacher: You have to obtain an important property of parabola. I can help you a bit: It is the property that Greek geometers obtained by examining this situation, where the parabola is already drawn. As you see, it is in three-dimensional space, on the surface of the cone. It is the same one that is described by the Cartesian equation of the parabola that you know. You have to discover the relationship between the green line segment [VS in Figure 2] and this line segment [PS in Figure 2].

1. The task

In this task, the conjecture phase is cut short: The students already know the property, which is expressed by the usual canonical form of the parabola equation. In other tasks (see Bartolini Bussi, 1993), the conjecture phase is explicitly assumed as a part of the task.

The teacher clearly states that the study in the solid setting and in the algebraic setting can be considered "the same": it is an example of regressive appropriation that depends on today's knowledge.

The small-group work can be divided into several episodes, which we have numbered and labelled. Some of these episodes contain joint activity with the teacher. Below are some exemplary short excerpts that are related to different issues:

a) the quality of help offered by the teacher to introduce the problem,
b) the quality of help offered by the teacher to explore the model,
c) the quality of dynamic exploration carried out by the students,
d) the quality of help offered by the teacher to sum up the whole process.

The excerpts have been chosen from the complete transcript to illustrate some critical features of joint activity. In particular, the issues (b) and (c) are related, as they represent some of the teacher's operations and the process thereby induced in the students. The effects of the help offered in (a) and (d) is better acknowledged in the final written report, which is analysed in the next section.

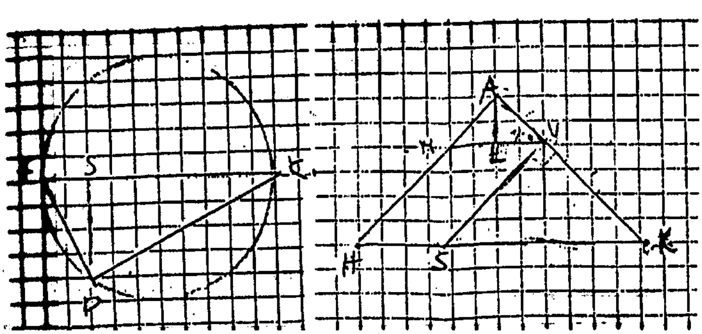

Figure 2: Figures produced by students during group work with coding letters

T: [...] The problem is to discover, on the basis of well-known theorems, theorems about right angles and similar right angles, which is the property that links these line

2. The teacher's introduction

In this long introduction, the teacher offers help by referring to some geometrical figures (right angles) to be focused on. He also segments, the green segment that we call abscissa and this segment that we call ordinate. You have to observe the model, this circle, this right-angled cone. You have to elicit all these hypothetical elements; then, by reasoning about them, you have to obtain a property of parabola that is perfectly equivalent to the equation that we write now, to the equation that Descartes would write many centuries later. It is the same relation, and it is possible to obtain it only in space. Look and try, say, for a quarter on an hour. It's a reasonable time. Then if you do not have any idea, call me; otherwise, go on with the things you have seen.

[...]

defines the degrees of freedom for the students: They cannot work for days but have a limited time to explore the model. At the same time, he introduces all the motives:

– the reference to history, to justify the accepted rules of behaviour
– the equivalence of synthetic and analytic description and their contribution to the meaning of parabola
– the necessary reference to the physical object so as to guess conjectures and be guided in proving

These elements will be refocused again and again during the small group to construct the sense of activity.

The students start to draw the fundamental elements of the model.

T: Sometimes it is necessary to consider also figures that are not immediately visible. For instance, in this case there is a right triangle that is fundamental, but it is not traced yet. You can look for it by taking into account that you need to consider those triangles whose sides are among the lines segments you have to relate to each other.
S 1: There is a right triangle [*OPS*]... It changes; when you change the plane the base is changed...
S 2: Yes, but in the meantime this other triangle changes...
T: This exploration seems a good idea. You have to reason in plane geometry but not always in the same plane; remember that Greek geometers saw the figure in space, even when they did plane reasoning, by considering different planes

[...]

3. Helping to explore
The teacher also gives methodological help to students to direct their search towards effective strategies. Students put into practice the teacher's suggestions: They have a rigid model, made of wood, Plexiglas, and thread; they cannot do real experiments (as in Cabri with dragging), but they see the changes. (The letters in brackets are only for the reader's help; see Figure 2.)

In this phase, the recourse to coding points by letters is very limited; the teacher too points at the model and speaks about "this triangle" and "this point". Coding with letters becomes essential (to the problem) only later, when proportions are to be written down. In this phase, it is more useful to build a dynamic ideal object on which to make a mental experiment for guessing conjectures.

S 2: Yes, this [$Y = PS$] is the height of the right triangle [*PHK*], *PS* the point [sic!]. *PS* is the height of the right triangle that is going to be formed [Italian: si viene a formare] in the semicircle. The angle in the semicircle is always 90°, isn't it?
S 1: It is! Hence this triangle...
S 3: Practically there is the triangle that

4. The dynamic exploration
The habit of moving a static object is typical of this classroom. I have explicitly stressed an unusual Italian expression (*si viene a formare*) that emphasises the progressive formation of an object that does not yet exist in the figure but is created at this moment in the mental process. Surely not all the

rotates on the semicircle; that is, it moves on the semicircle.
S 2: Only the height of this triangle is changing [she gestures with her hand palm upwards to show that when the horizontal plane is going up the height of the triangle *PHK* is changing].
S 3: Considering the two planes [i.e., the horizontal and the oblique plane]... Yes, but there is also this plane [the vertical plane].
S 2: In this plane [vertical], if we consider this triangle [*VAS*] when the plane changes, when the plane moves, it changes too.

explorations are effective for the solution; they are indeed necessary to create the dynamic ideal object.

The process of building a proof is long. It is necessary to choose the useful triangles and to mark on them the useful proportions. The proportions are to be interpreted within the theory of application of areas (*to behave like Greek geometers*), up to the statement of the *symptom* of parabola:

$$2\ VA\ VK = PS^2$$

Only later does the teacher suggest relating this formula to the post–Cartesian approach. It is done by making the following substitutions:

$$VK = VS = x \qquad PS = y \qquad VA = k$$

This yields:

$$2\,k\,x = y^2$$

The final interpretation is done by the teacher, who sums up all that has been already said during the group work:

T: When we write

$$2\ VA\ VK = PS^2$$

or

$$2\,k\,x = y^2$$

it is the same. Only the notations are changed. This is the modern notation, and that is the one that they [the Greeks] used, but the equation is the same: It is the equation of the parabola. It is the geometric property of parabola.

That [i.e., *VA*] is constant, since when I move this one [he gestures to indicate moving up and down the horizontal plane], this does not change, because the vertex of the cone and the secant [oblique] plane are

5. Helping to sum up the process
At first, the relation between the proportion and the equation is recalled. This is a way to emphasise the spatial interpretation of the equation of the parabola that otherwise risks being lost.

The introduction of the constant k is justified on the model by observing that a part of the configuration does not change when the auxiliary horizontal plane is changed.

The change in the secant plane is *imagined* on the fixed model by gesturing.

the same.

If I cut with a plane closer to the vertex, how does the parabola change?

Because, if I keep the secant plane fixed and change this [the horizontal] plane, the parabola is the same; what changes is only this arch that becomes longer or shorter. But if I change the secant plane, and I cut the cone with a secant plane closer to the vertex of parabola, how does the parabola change?

Ss: [looking at the model] It narrows.

T: Right, it narrows. And if this becomes longer, if this plane goes here...

Ss: It widens.

T: Right. And this is what happens [he points at the equation], isn't it?

Let the equation of the parabola be

$$x = h\,y^2$$

If this [h] changes, the width of the parabola changes.

The secant plane goes closer to the vertex...

...or farther from the vertex...

...and this change is related to the equation.

2.3.8 The final report

The report is produced as a collective homework by the group after the laboratory ends, on the basis of drawings and personal notes. In the left column below, the students' text (before any correction by the teacher) is translated literally. In the right column, a division into sections is suggested. This division clearly shows that the students have always succeeded in giving a logical organisation to the long text. The complete analysis of the transcript (Tufo, 1995) shows that the order is not the same order of the exploration during the group work: Then it represented a further control on the collective activity.

Plane Section of a Right-Angled Cone

It is necessary to distinguish between RIGHT cone and RIGHT-ANGLED cone. On the one hand, a right cone is when the perpendicular from the vertex to the plane of the circle (i.e., the directrix of the cone) is in the centre of the circle itself. On the other hand, a right-angled cone is generated by rotating an isosceles right-angled triangle about a cathetus [leg]. The cone is called right-angled as the angle between two opposite generatrices is 90 degrees. If it were acute, the cone would be acute-angled; if obtuse, obtuse-angled.

1. Definition of right-angled right cone

The students start from a possible misunderstanding about right and right-angled cone. They probably remember a personal experience. They recall Euclid's definition and relate the definition of right-angled cone to other kinds of cones. This means relating the orthotome to other conic sections.

They are consciously in the solid setting and are consciously using an approach inspired by history. A few lines below, they make an explicit reference.

Description
The model reproduces a RIGHT-ANGLED cone generated through the rotation of an isosceles right triangle cut by a plane perpendicular to the generatrix *AH* and parallel to the opposite generatrix *AK*. The model contains also two perpendicular PLANES: the plane *t* of the circle (directrix of the cone) and the plane *t'* (MERIDIAN) of the axis *AO* of the cone. The plane *t'* contains the segment *VS*, from the vertex of the section (ORTHOTOME) to the point *S* (the intersection of the diameter *HK* of the circle of the plane *t* with the line *PM* of the secant plane).

The section of a right-angled cone produces a curve named ORTHOTOME by the ancient geometers prior to Apollonius.

2. Description of the model
The physical object is carefully described. Two planes are explicitly named: The former [*t*] is a physical plane, realised by a wooden base; the latter [*t'*] is an ideal plane, determined by a wooden frame. The secant plane is made of plexiglas.

The reference to history is explicit.

The property or SYMPTOM (verified by all the points of the curve), that allows one to recognise the kind of plane section of the cone is based on the ~~equality~~ [equality is erased] relationship between the segments *PS* and *VS*, i.e., on the ~~equality~~ [equality is erased] equivalence of two geometrical figures that individualise the position of the point P.

3. Task
The students are recalling the task and the accepted rules of behaviour that have been stated by the teacher in the contract. They have to behave like Greek geometers and use proportions and equivalence of areas. An incorrect term is erased by them and replaced by a better one.

The reasoning from which the property is drawn, a property valid not for a special point of the orthotome, as the secant plane maintains always the same distance from the vertex of the cone and the same slope, is the following:

(Generalisation to any point of the orthotome)
Someone might believe that the reasoning works only for the special point of the figure: The students explain that it works for every point, anticipating now what will be argued at the end (see point 9 below).

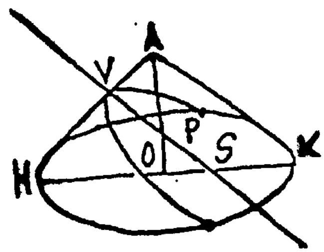

Figure 3: The first drawing produced by the students

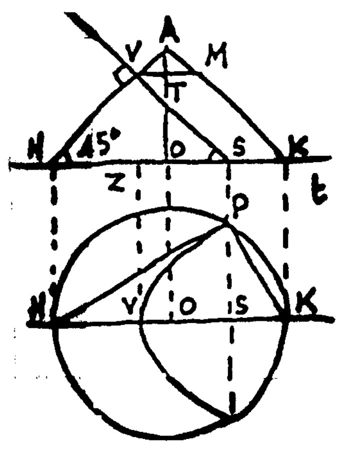

Figure 4: The second drawing produced by the students

Let us apply the reasoning first to the plane t on which the circle is.

If we consider point *P* of the orthotome, we observe that it, obtained from the intersection between the secant plane and the directrix of the cone, lies on the circle, whatever the distance *AV* between the vertex of the cone and the plane. Every triangle inscribed in a semicircle has an angle of 90°; hence the triangle *HPK* inscribed in the semicircle with diameter *HK* is right-angled, with HPK = 90° (THEOREM). Hence it is possible to apply the *theorem of Euclid* in

4. Reasoning on the plane *t*
The students are using the theorem of Euclid for right triangles.

They state the theorem referring to mean proportionals...

right-angled triangles. The height PS, drawn by means of the perpendicular from P to the diameter HK, is the mean proportional between the projections of the catheti [legs] of the triangle on the hypotenuse, i.e. the diameter HK.

We state the proportion ($HPK = 90°$):

$$HS : PS = PS : SK$$

that is,

$$PS^2 = HS \cdot SK$$

We have proved that the area of the square built on the segment PS is equal to the area of the rectangle with sides KS and HS (geometrical interpretation).

...and interpret the proportion, like Greek geometers, as an equivalence of areas.

Let us consider now the meridian plane t' perpendicular to the former one. We try to state a proportion that can relate PS and VS. We observe that on the meridian plane there are two similar triangles. The former is HVS, with hypotenuse the projection of the cathetus PH on the diameter HK and cathetus the distance from the vertex of the orthotome to the intersection of the generatrix AH with the circle. The two catheti VS and VH are equal, as the angle VSZ is 45° (the secant plane is perpendicular to the generatrix AH, and the angle VHZ is 45° (AHO is an isosceles right triangle.

Hence the triangles VZH and VZS are equal, whence $VS = VH$.

The other right triangle to be considered is AVT, formed by the line VM parallel to the diameter HK and by the axis AO of the cone.

As the angles $TAM = VAT = 45°$, $HAK = 90°$. As they both have an angle of 90°, $AVT = AMT = 45°$ and $VT = AT$.

The triangle AVT, as an isosceles right triangle, is similar to the triangle HVS. We can state the proportions:

$$HS : VH = AV : VT$$

hypotenuse : cathetus = hypotenuse : cathetus

That is,

5. Reasoning on the plane t'

Among the many triangles and proportions that could be observed in the model, the students claim to focus on a relationship between PS and VS. This statement clearly represents a conscious control of the strategy. In the workgroup, explorations have been actually much more extensive: not all the explorations have proved to be useful.

The two isosceles triangles are found. The proportion is stated with the control of the meaning...

...and translated into an equivalence of areas.

$$AV \cdot VH = HS \cdot VT$$

If we compare the proportion with the above relationship

$$PS \cdot PS = HS \cdot SK$$

we observe that

$$HS = 2 \cdot VT$$

as $VS \parallel MK$, i.e., $VMSK$ is a parallelogram and the triangles

$$AVT = ATM.$$

We obtain:

$$SK = 2 \cdot VT$$

$$PS \cdot PS = HS \cdot SK.$$

As in both the first and the second equations [sic], there are both HS and SK

$$2\,AV \cdot VH = 2\,HS \cdot VT$$

multiplying each member by two in order to obtain

$$2\,VT = SK$$

whence

$$PS^2 = 2\,AV \cdot VH.$$

The area of the square built on the height PS is equal to twice the area of the rectangle with sides $VH = VS$ and AV.

6. Linking the two planes together
Up to now, two different steps have been realised: the former in the base plane t, the latter in the meridian plane t'. A link between them can be found.

A fundamental relation is obtained (the *symptom*) and interpreted as equivalence of areas.

If we introduce a suitable system of coordinates x and y in the secant plane, the coordinates of P are given by

$$x = VS \text{ and } y = PS.$$

Hence, we have

$$y^2 = 2\,AVx.$$

As in the reasoning the distance between V and A is always the same, AV = k
then $y^2 = 2kx$ and $x = (1/2)k \cdot y^2$.

7. The system of coordinates: from symptoms to equations
A conscious anachronism is introduced to shift from the solid to the algebraic setting, and the standard equation is obtained.

The k constant is introduced by repeating the teacher's words (see "Helping to sum up the process" above)

When k changes, i.e., when the secant plane is translated parallel, the width of the orthotome increases. On the other hand, when k decreases, the width decreases until, when k decreases to zero, we obtain the degenerate orthotome, i.e., the line AK (equation $y = 0$).

8. The meaning of k
The meaning of k is constructed by changing it. Also, in the change, the limit case appears and is interpreted correctly.

The obtained symptom characterises whatever point of the section is chosen because, if the distance $AV = k$ is the same and the plane t (where the generator circle lies) is translated parallel to itself, the proof is valid for any other point on the section. Only the length of the arc of the curve has changed.

9. Generalisation to any point of the orthotome
The reasoning is generalised to any point of the orthotome by moving one of the fixed planes.
Actually, in the movement some lengths are not changed. The final sentence shows the correct interpretation on the model.

Such a section was considered as a solid curve in three dimensions, as it lies on a right-angled cone, and its property is obtained by means of reasoning about two different planes that are perpendicular in space.

10. General comments
This final comment shows the detachment of the students from the approach of Greek geometers. The section *was* considered a solid curve. Actually, this term also appears in Descartes' work. This conceptualisation is correctly related by the students to two different issues: the way of obtaining the curve (as a conic section) and the way of proving the symptom by means of figures lying in two perpendicular planes.

3. DISCUSSION

In this section, I shall go through the teaching experiment once more to summarise the relationships between the motives, tasks and operations as they can be detected by looking at the teacher's side, in both the designing and the functioning, and their traces that can be detected by looking at the students' side, in both the oral interaction and the written report. A brief comparison between teacher's side and students' side can be seen in Table 1. This analysis is surely incomplete since I shall refer only to the short excerpts that have been quoted in the text. Some reference to the other available data (Tufo, 1995), however, will be made from time to time.

3.1 The teacher's side

In the section "Motives of the teaching-learning activity for Mathematical Machines" the four motives are listed. The first motive is realised by means of extensive historical introductions, with the reading of original sources, too. The meaning of mathematical concepts cannot be constructed only inside mathematics, so the problem of the *construction of meaning of conics* in the classroom has to be taken as paradigmatic and as representative of different approaches to the problems of space and geometry in the different cultural spaces of different ages (a similar approach can be found in Mancini Proia & Menghini, 1984). The second section of this chapter sketches the kind of historical introductions that are proposed by the teacher. They are not limited to internal history of mathematics but they allude to the wide social and cultural environment where mathematicians of the past lived. Moreover they stress the collective aspects of scientific progress, where to locate individual contributions: hence Euclid, Descartes, Desargues and others are not conceived as isolated geniuses that work in a vacuum, but as exceptional representatives of existing cultural trends.

In this teaching experiment this first motive is especially focused in the initial historical introduction (section “The teacher's lesson: A historical introduction to conics in ancient Greece” and in the presentation of de l'Hospital's work (section “The rest of the experiment”).

The second motive contributes to defining the rules of the didactic contract: in some tasks students are allowed, like Greek geometers, to use application of areas and proportions and forbidden to use algebra; in other tasks students are allowed, like post-cartesian geometers, to introduce a system of coordinates on the figure and to write down algebraically the relationships between some line measures; in other tasks only the algebraic equation is considered and so on. The rules are explicitly posed by the teacher at the beginning and recalled during the interaction. One of the effects is that different formats of proof of the “same” statement are considered, so that the meaning of the statement is enriched by the whole activity. The social rules are explicitly related by the teacher to the issues that have been presented in the general historical introductions.

In this teaching experiment, this motive is focused in all the phases of the study of orthotome (sections “Small-group work: The study of the orthotome model”, “Students' explanation of the orthotome model” and “The teacher's lesson: Models of the orthotome and the equations of conics”). Some traces are found also in the excerpts we have quoted, at the level of teacher's operations. For instance in the Episodes 1 and 2 of small group interaction, the teacher emphasises the difference between “then” (ancient Greece) and “now” (after Descartes) and clearly states the first task to be solved without coordinates. Later, in the Episode 5, he states again the relationship between ancient and “modern” notation.

The third motive is realised by the sequence of actions in the teaching experiment, with intentional shift to and fro the solid setting and the algebraic setting (and later, with De l'Hospital's work, the mechanical setting too). In the Episodes 1 2 and 5, explicit relationships between the two settings as concerns the description of the curve and the instrument of proving are stated.

The fourth motive is realised by the systematic and intentional recourse to physical models, either static or dynamic ones. Traces of this emphasis are found also in the interaction: in the Episode 2 the teacher explicitly invites students to observe the model and to elicit the hypothetical elements. In the Episode 3, the teacher encourages students to make exploration: he does not use letters for coding points, but points to the model with gestures; he suggest to introduce also not visible elements and to modify in the mind the physical object. In the Episode 5, he shows that the point P can be considered a "generic" point, by introducing movement; he interprets the change of the secant plane both in the model, imagining a movement, and in the equation, imagining a continuous change in the numerical value.

TEACHER'S SIDE			**THE STUDENTS' SIDE**		
Motives/ Activity	**Actions (examples)**	**Operations (Examples from Small Group Work)**	**Traces in Small Group Work**	**Traces in Final Report**	**Elements of Constructed Meaning**
Relationships between mathematics and other fields of knowledge	Wide historical introductions	Not detailed	—	—	Historical contextuali-sation
Historical contextuali-sation of rules of behaviour	Definition of didactic contract	In Episodes 1, 2 and 5	Not detailed in the excerpts	Sections 1, 2, 3, 7and 10	Of problems concepts and procedures
Multifaceted meaning of conics	Shift to and fro solid, algebraic (and mechanic-al) settings	In Episodes 1, 2 and 5	Not detailed in the excerpts	Sections 7, 8 and 10	Interplay between settings
Dynamic interpretation Principle of continuity	Systematic recourse to physical models	In Episodes 2, 3 and 5	Gesturing Episode 4	Section 8 Section 9	Dynamic interpretation of physical models

Table 1: Comparison between teacher's side and students' side

3.2 *The students' side*

Now we shall go through the students' protocols to find traces of motives, if any. We shall draw on a very limited set of data, only a couple of excerpts from interaction and the written final report. However, the final report is very long and interesting, because it contains, in a well-ordered style, all the relevant issues of small group interaction. The order of the text is the first relevant feature. This order does not mirror the complex exploration of the model during small group work: having been able to contextualise and to write down the proof on the base of sketchy notes (taken by students standing close to the physical model and not sitting quietly at their desks) proves that the text conveys the constructed meaning.

Traces of the first motive cannot be acknowledged in this limited set of data. Actually changes in the conceptualisation of mathematics are not always explicitly stated by students. They can be revealed by long term listening at their talks; for instance they are evident in the quality of discourse they produce in oral test, in mathematics and in other subjects as well (e.g. history, philosophy, literature and so on).

On the contrary, traces of the second motive are evident. In the Sections 1, 2, 3, 7 and 10 of the written report, explicit reference to history is done again and again. Actually this reference is functionally interlaced with proofs, in the solid setting and in the algebraic setting as well.

Also traces of the third motive are present. In the Sections 7, 8 and 10, conscious anachronism is repeatedly commented: the students are aware that today we can consider the conic section orthotome and the algebraic curve parabola as the same object, but they are conscious that both the definition and the instruments of proof are quite different and historically contextualised.

The fourth motive is revealed by different sets of data: the large amount of gesturing in small group interaction (some examples are in the Episode 4 of small group interaction); the conscious recourse to movement in interpreting the meaning of k (Section 8 of the written report) up to the analysis of the limit case ($k = 0$) that had not been considered during small group interaction; and the conscious recourse to movement in extending the property to any point of the curve (Section 9 of the written report). Actually some of these points had been hinted at by the teacher very quickly: the independent written reconstruction is so clear and neat that the students are supposed to have internalised joint activity with the teacher.

If this analysis is correct, the meaning of conics that is now constructed by students is more complex than in standard classrooms. At least three elements that are usually lacking enter as constitutive parts of the meaning:

1. the historical contextualisation of problems and concepts,
2. the relationships between the solid setting and the algebraic setting (and, in the last step, the mechanical setting),
3. the dynamic interpretation of physical models to guess conjectures and to guide the construction of proofs

The development of this complex meaning draws on two special choices that have been made in designing the teaching experiments: the recourse to historical sources and the activity on physical models.

As concerns the former, it is clear from the text that, at the beginning, the students are (consciously) in the solid setting, while later they shift (consciously) to the algebraic setting. The interplay between the two settings is evident when they interpret the change of the parameter k (up to the limit case $k = 0$) as a parallel translation of the plane. Hence the model is studied with the introduction of a conscious anachronism, that shows limits and advantages of each approach. In the process of solution of the given problem, there is a shift from one setting to another and from one time period to another; there is a continuous change of the objects, that are identified by means of a process of regressive appropriation: what they know now on conics allow them to go to and fro the individual sections and the individual settings, using the most advantageous tools for proving. In this process the historical reconstruction of meaning is not a way to motivate students or to embellish the problem but is a fundamental object of the teaching learning activity.

The presence of the physical model is essential in the process of solution. The observation of the small group work (during which the proof has been built) has shown a large volume of visual tactile activity (e.g., gesturing, pointing at the model) while the discussion was going on. This is an invariant aspect of all the laboratory sessions. Yet a transformation of the physical object into an ideal object is observable, as the study of the physical object is done with reference to Euclidean theory of proportions and to analytic geometry. This process between the physical object and the ideal object is dialectical: at the end the interpretation of the values of the parameter k is done on the physical object; besides the generalisation of the properties to a whichever point of the section is done again on the physical object. The translation of the planes (the plane of the section for k, and the base plane for the generalisation) is done looking at the physical object (where the plane are fixed) and moving the hands up and down (a further analysis of a similar process on a linkage is done in Bartolini Bussi, 1993).

3.3 Open problems

In this paper we have introduced some elements of an historical analysis of conics (to be meant as a paradigmatic example of geometrical concepts) to claim that their present meaning as objects of the knowledge to be taught is not one-sided but ground in different settings determined by the processes of studying conics in different time periods.

The didactic problem is how to introduce in the classroom the epistemological complexity suggested by this historical study. A pragmatic solution is offered by the research project *Mathematical Machines*. But this solution opens two different kinds of problems, concerning:

1. the microstructure of teaching-learning activity in the classroom,
2. the relationships between the elements of the context (mainly historical sources and physical models) and the student processes.

The former problem has to be meant as the tentative modelisation of the teaching-learning activity at the level of actions-operations (Leont'ev, 1978). As we adopt a Vygotskian perspective on the teaching learning process, we claim the need of considering phases of joint activity between the teacher and the students (Bartolini

Bussi, 1993); however the complexity of the processes does not allow us to make a priori analysis of them.

The latter problem concerns the two main distinctive characters of the context, that makes it different from the ordinary one, namely the systematic introduction of physical models and of historical sources. The data we have collected until now show the extraordinary effect they can have on students' construction of meaning. We are interested in detecting more precisely the roles and the conditions of functioning of both in this process. Physical models emphasise the role of visual tactile activity in a way that appears quite different from software tools: the stiffness of physical models forces students to make experiments in the mind and to anticipate results that cannot be controlled empirically, while the flexibility of dynamic software such as Cabri rather invites students to make concrete experiments and to observe their effects. Historical sources fosters student self location in the collective cultural activity of mankind. Both aspects are studied in the activity theoretical approach drawn on the work of Vygotskij, Leont'ev and others (see for instance Tikhomirov, 1984 for the former; Otte & Seeger, 1994 for the latter). So, the main aim of our research group now is to look for a comprehensive theoretical framework that allows us to interpret the relationships between these two characters and student construction of meaning and to design further teaching experiment where to realise and analyse such relationships (Bartolini Bussi et al., forthcoming).

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