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# THE MATHEMATICAL ENCYCLOPAEDIA OF THE 15$^{TH}$ CENTURY

## Arithmetical operations, fractions, progressions, linear equations and roots of real numbers, according to the Codex Vindobonensis phil. gr. 65 of the 15$^{th}$ century.

**Summary.** I present some few results of my study on the mathematical content of the published part (f. 11r–126r) [8] of the Codex Vindobonensis phil. Graecus 65 (Tractatus Mathematicus Vindobonensis Graecus or TractMathVindGr) which the author is anonymous. The other part f. 126v-140r. has been published by H. Hunger and K. Vogel in 1963. This 15$^{th}$ century (1436) Byzantine MS includes the solution of problems of practical arithmetic, and algebra, the roots of which can be traced back to antiquity and their comparison with modern mathematical methods and terminology [9] reveals -apart from some differences- many identities and similarities showing the unbroken continuity of mathematical tradition through the centuries.

## The arithmetical operations and their checks

The symbols, which are used in TractMathVindGr are the letters of the Greek alphabet but the calculations are carried out with the new decimal Hindu-Arabic system of numeration. Even though the author is not used to the new symbolisation, it should be emphasised that the use of letters and not numbers does not affect the result, since it concerns a system in which the arithmetical value of a letter depends upon its place [62]. Thus, the author of this manuscript insisted on preservation of the old symbols, whilst other earlier scholars, such as Maximos Planudes (1255–1305 A.D.) in Byzantium and Fibonacci (born in 1170), who introduced the new arithmetical symbols in Western Europe, were familiar with the new symbolisation and the new arithmetical system. In his work of ‘Liber abacci’, Fibonacci uses the new numbers, and Planudes did the same in his work [25].

However, the use of the new symbolisation was not generalized during the Byzantine period. We know indeed that eminent Byzantine scholars, such as Pachymeris, Moschopoulos, Rabdas, Johαnos Pediasimos, as well as Barlaam Kalavros, Isaac Argyros (14$^{th}$ c.) [63] did not use it. It is possible that the author did not adopt the new numbers because their use created various problems in commercial mathematics. In 1299 A.D. the Mayor of Florence issued an order, according to which the writing of numbers in columns was prohibited as well as the use of Hindu numerals, because they could easily be changed and the zero become 6 or 9, a danger which did not exist using Roman numerals. Also a guide which was published in 1594 warned the merchants that they must not use the new arithmetical numerals in their contracts and transactions [67].

In the codex 65, the term “milliouni” is mentioned which means a million as it appears from its definition. We know, of course, that M. Planudes was one of the first who used the term “milleton” (e.g. a million) for the million. According to D. E. Smith, this term first appeared in 1478 in the Italian manuscript “Arithmetic of Treviso”. We note that in this manuscript, which is subsequent to Codex 65, in the act of multiplication, the number which multiplies is made descending downwards from the number which must be multiplied, and this is done in a similar manner to that used in Codex 65 [40]. We therefore have an important indication that the term “milliouni” did not first appear in the Italian “Arithmetic of Treviso” but in Codex 65, which appears to date back to 1436 A.D. We note that the year 1436 had maybe been determined somewhat without authorisation at to the end of the Mediaeval period (529 A.D.-1436 A.D.) [6].

In 1494 Luca Pacioli issued the “Suma” [37]. “Summa de Arithmetica, Geometria, Proportioni et Proportionalita” of Luca Pacioli was the first mathematical encyclopaedia of the Renaissance. The first part includes Arithmetic and Algebra and the second part Geometry, exactly as in the codex 65. As sources, Luca Pacioli used Fibonacci, Jordanus Nemorarius, Vlassio of Parma, Prosdocimo de Beldomandi, and al Khwarizmi. Pacioli used the Hindu numerals [68] in “Suma” and calls the “crosswise method” of multiplication “crocetta” (little cross). For example in multiplication of 12 with 13, initially the 2 was multiplied with 3 to make 6. Further, the “crosswise” digits of 12 and 13 were multiplied and the results of these multiplications are added, so we have 1.3 + 1.2 = 5. The 5 represents the decades and the 6 the units. Further multiplication of the first digits of the numbers 12 and 13 arrives at 1. The 1 represents the hundreds and thus the result of multiplication of 12 with 13 is the number 156.

The term “multiply crosswise” was used in the analogies of the form $a/\beta = \gamma/\delta$ from which we arrive at the equivalent of $\alpha.\delta = \beta.\gamma$. Today in similar circumstances, we use the term “multiply diagonally” [29]. In the same work, Pacioli who taught arithmetic and commercial algebra mentions to the method of “four-sided” in multiplication of two 3-digit numbers, in which the number which multiplies is made descending downwards from the number which must be multiplied. However this is exactly how multiplication of three digit numbers is done in Codex 65, which is older than the “Suma” [41]. The similarities of this Codex in relation with the “Suma” and with the “Arithmetic of Treviso” do not stop here since in the second one, the division is done in a similar way to that of Codex 65 [42] .

Of course, the interactions between the Byzantines and Western are undoubted since Planudes makes division using the Fibonacci method, which is also identical with the method used in Codex 65. Planudes clarifies that it concerns laborious work, a view that was shared by his other counterparts [43].

The method of checking multiplication [10] which is based on a rule which is based on the remains of the divisions with the number 7 is too old. With this method the multiplication of 15 by 6 is done as follows:

αε γ
ς
θυ

To test this multiplication the author requires the remainder of the division of 15 by 7, which is 1. Because the remainder of the division of 6 by 7 is 6, multiplication of 1 with 6 placing the remainder in a circle. Finally the remainder is found in division of 90 by 7, which is 6, to be compared with the number, which has been placed in a circle. Since the two results are the same, then the multiplication is correct.

The Hindu used that method, by dividing by 9 instead of 7. Al Khwarizmi (c. 825 A. D.) was familiar with this method as well as Al Karkhi (c. 1020 A.D.), who are even more ancient than the actual date of Codex 65. We consider then that this indicates the possibility of the origin of this method from the Hindus. We also know that the Arabs had adopted this method, using of course the number 7, as well as 8, 9 and 11. It appears that the Rabbi ben Ezra (c. 1140) was influenced by them as well as Johannes Hispalensis (c. 1140) Fibonacci (1202 and Maximos Planudes (c. 1340) [20] and Pellos (c. 1492) who writes that the check by

7 ensures a very little possibility of error [44]. The same opinion was expressed by the author of Codex 65.

Hippolytos (200 A.D.) in his work ‘‘Kata passon airesseon elenchos’’ (IV– C. 14, Patrologia Graeca, ed. Migne, vol. 16, 3, line 3078), sets a problem regarding the discovery of the “base” (o pythmin) of the name Agamemnon. The sum of the bases (base = the remainder of the division of the number which corresponds to each letter by 9), is equal to 36. Because $3 + 6 = 9$ the base of the name is 9. If the sum was 19, then as Hippolytos wrote the base would have been 1 (because $1 + 9 = 10$, therefore $1 + 0 = 1$). In a secondary manner he divided by 7 and not by 9.

This method is based on the following theory: “The remainder of the division of a number by 9 or 3 is the same as the remain which results if the sum of the digits of the number is divided by 9 or 3”. This theory is not found in the ‘‘Elements’’ of Euclid*,* because it belongs to the area of practical mathematics [58].

As far as the Hindu or Arabic origin of this method is concerned, we should point out that the scientific knowledge of the Arabs particularly in algebra and astronomy, was rather an inheritance from the Greeks and did not originate from them [59]. The 9th century was for the Arabs that of translation of Greek manuscripts. They knew works, which have not survived today, and because the Arabs were not interested in the return of ancient Greek knowledge to the Byzantines, slowly the Greek manuscripts disappeared, and the Arabic prevalence followed [33]. Although this method is not in use any more, we found it in a 20th century’s book, i. e. ‘‘A detailed description of Theoretical Arithmetic for Practical Schools’’ of Secondary Education written by N. Nikolaou, which taught in the fifties [36]. This does not mean of course, that the aforementioned method was taught up to that time continuously at all schools. Immediately after the fall of Constantinople, the lower schools taught the ‘‘Arithmetic’’ written by Emmanuel Glyzonios for more than two and a half centuries. In this Arithmetic*,* the check of multiplication was done by the crosswise method [16].

Another field, to which the author of Codex 65 pays particular attention, is how to carry out operations without pencil and paper, i.e. from memory (“in the imagination”). His operations are based on algebra, which was still “rethoric” as symbols were not then in use. Namely for multiplication of 13 by 13 the author uses the following procedure:

Multiply ι by ι results in ρ. Add γ to γ result ς. Add ξ to ρ and θ results as ρξθ. These are explained today in a scientific speciality e.g. $(10 + 3)\ (10 + 3) = 100 + 6x\ 10 + 9 = 169$.

Later within the years, when Cardano (1501–1576 A. D.) issued the *Practica Arithmeticae* (1539) he showed the same capability of regarding calculations from memory [19].

## Fractions and their operations

In TractMathVindGr the way of defining a fraction is based on the condition that the numerator must be smaller than the denominator. The same notion is extended, within the same Codex to fractions with numerators greater than denominators. The most unusual thing is that in the Arithmetic of Pagani (1591 A. D.) the numerator is less than the denominator, whilst fraction with numerator greater than denominator is considered to be a subsequent discovery [45].

In Codex 65 the operations between fractions are carried out using methods similar to those of today. This is another indication of the unbroken tradition of mathematical methods until today [26]. The author uses the term “ὀκτακαιδέκατον” as well as other terms related to it, in order to denote 1/18 and other similar fractions. At this point what is clear is the influence from antiquity, meaning that of Heron of Alexandria, up to George Pachymeris, while both were using the same terms [61].

Concerning the rule of three, which is still used today, as to its origin and name, this is considered to be Hindu, and that it was adopted later by the Arabs and the Latins. The method of three was particularly popular in commerce, and it was called “Golden rule” or “Rule of commerce” or “Merchants’ key”. It is possible that the origin of this method comes from Brahmagupta (628 A. D.) whilst Sridara ($10^{th}$ c. A.D.) wrote about this method, which he used in his work, as later did Bhaskara (1150 A.D.) [46]. In Codex 65 this method is frequently used under the name “treatment by the three” (chapter 53, f. 34v) on the basis of the peculiarities of the proportions (chapter 55, f. 35r). In the chapter 53 the author writes: “the answer is found through the numbers γ and δ and ε which are three unequal numbers”.

Today we also know that according to Leibnitz (1674) “The analogy is based on the fact that that which agrees with many things or that which is opposite, also agrees with the neighbouring of the nearest things with which it is found in the opposite side”. According to Leibnitz, the fact that the same rules which apply in finite objects apply to the infinite is based on the proportions. Leibnitz who also was based on Euclid axiom, that the whole is greater than its parts, concluded that there is no number which is greater than all the others. Later, Dedekind modified the authentic opposition to a term of the infinite, namely: “A total is infinite, if and only if, there is a bijection amongst this and its authentic sub- total” [32].

The rule of three was a marvelous method of teaching for teachers in those days and even earlier [27] of the peculiarities of ratios which had been known since antiquity [15], applying them to problems of everyday life which interested people who had no theoretical training e.g. the merchants, the technicians and others. It is noteworthy that in a book of Arithmetic dating in the $16^{th}$ century the name “Reoules” used to denote apart from the method of three, the method of five and seven, namely the compound (syntheti) method [17].

## Arithmetical progressions

The author of TractMathVindGr works on progressions, particularly with various types of arithmetical ones for which he recommends the group methods of solutions. Although the term “arithmetical progression” belongs to Diophantus [14], whose work copied and commented upon it Maximus Planudes ($13^{th}$ cen.) [13], the author of Codex 65 does not name the sums “arithmetical progression”. Al Karhi [1] got involved in these sums in the $6^{th}$ c. A.D. and Avicenna in the $11^{th}$ c., A.D. who calculated those sums by applying the following method [2].

$$1 + 2 = 3 = 2 + (1/2) \cdot 2$$
$$1 + 2 + 3 = 6 = 2 \cdot 3$$
$$1 + 2 + 3 + 4 = 10 = 2 \cdot 4 + (1/2) \cdot 4$$
$$1 + 2 + 3 + 4 + 5 = 15 = 3 \cdot 5$$
$$1 + 2 + 3 + 4 + 5 + 6 = 21 = 3 \cdot 6 + (1/2) \cdot 6$$

The edition of ‘‘Arithmetical Introduction’’ by Nicomachus Geraseni [18] ($2^{nd}$ c. A.D.) contains some problems attributed to a monk called Isaac, which refer to the calculation of the terms of an arithmetical progression. These are problems similar to those in Codex 65, the answers to which are the same as those to be found in Codex 65 [60].

It seems that no general rule was discovered for calculation of all forms of arithmetical progression, as is the case today. This may be due to the fact that algebra was then still descriptive (without symbols). They could then have known the formula which gives the sum of n first terms of an arithmetical progression, as well as the formula which gives its n-th term, but they did not however recommend their application, because, the students would have been led to a greater number of operations which they had to remember how to use them solving a problem of this type. For example, in the Codex 65 for the calculation of the sum $\alpha + \gamma + \varepsilon + ... + \iota\zeta$ the additions are added one by one and makes $\pi\alpha$. The author of TractMathVindGr also states the general method, according to which: $\iota\zeta = \eta + \theta$, $\theta \cdot \theta = \pi\alpha$ and this is the sum sought.

We know that this concerns arithmetical progression with the first term being 1, ratio 2, and the last term being 17. As a result, applying the type $\alpha_n = \alpha_1 + (n - 1)\,\omega$ we have $17 = 1 + (n - 1) \cdot 2$, e.g. $16 = (n - 1) \cdot 2$ and $n = 9$. The sum sought will be

$$\Sigma_n = [(\alpha_1 + \alpha_n)/2], \text{ e.g. } (1/2 + 17/2) \cdot 9 = (18/2) \cdot 9 = 81.$$

## Problems of linear equations

In this chapter the author deals with problems, which are easily solved today by using linear equations, despite the fact that he himself however solves them with practical arithmetic. Because, the author teaches in the following chapters, the methods of solution of quadratic, cubic and biquadratic equations, anyone would have expected him to solve these problems using equations, so that his students would have had kept less operations in memory.

A customary method used at that time was the one of “false assumption”. Although the use of this method of “false assumption” leads the author, as is to be expected, to a false conclusion result, he reaches the correct answer by applying the qualities of proportions [69].

As is well known, the problems of equations of first order have there roots in antiquity [47]. Although they are initially problems of algebra, they were solved by using practical arithmetic. It is worthwhile noting, that these problems were found in Arithmetic books which were considered more advanced than the usual ones [48]. This indicates that Codex 65 was probably a worthy Arithmetic of its time.

The method of “false assumption” was particularly beloved by Diophantus, who used it to solve linear equations, the result of which he used to find by comparison [21]. This very ancient method was taught at schools in Europe and America up to the 19th century. It seems that this method was very well known in Medeaeval times since Fibonacci related to it in his works [49] and used it often in problems [64].

We emphasize that most of the problems, which are solved using linear equations, appeared as samples of a type of enigma. According to Smith, these problems have Greek origins and of course, many of them attribute to Metrodoros, but they also got influenced by the East. Later Rabbi ben Ezra (1140) and Jordanus Nemorarius (1225) and other scholars got involved with them [50].

Another type of problem relates to movements for meeting or removal of ships or persons. The author of Codex 65 does not consider them as real problems (chapter 71, f. 40v) but only a type of exercise for students, so to be able to deal with the consequent subject in question [11].

Metrodoros is considered as the main creator of these problems, which belong to recreational mathematics. Problems relating to the motion of persons and ships may be found in the ‘‘Liber Abbaci’’ of Fibonacci [65]. It seems that theese problems have Chinese roots and, as Smith asserts [51], they first appeared in the West in 1483 and were found in the manuscript ‘‘Suma’’ of Luca Pacioli, written in 1494. If Smith’s assertions are correct, it is very likely that Codex 65 is the source from which Pacioli drew subjects, when he wrote his *Suma*. If we take under consideration that the Arithmetic of Treviso, which was written in 1478, was also based on the *Suma*, it is highly likely that the contents of Codex 65 would have some particular meaning in relation to the teaching of mathematics in those days. The question therefore arises, concerning the relationship of Codex 65 with the other two manuscripts, since the Arithmetic of Treviso, which is anonymous, is considered to be the first commercial arithmetic of those times [30].

It is worth noting that Pacioli's Suma, was taught until the 16$^{th}$ century and was considered to be a mathematical encyclopaedia [31]. Of course, this piece of work was not known for new discoveries in mathematics. However it gives us information about the mathematical knowledge up to its time and is considered that it laid the foundations for the further development of algebra in the 16$^{th}$ century. The author of Arithmetic of Treviso, who came from Venice, would like to assist the merchants and his cleric friends. For this purpose, his work contained problems of the four operations (multiplication was done "crosswise" and calendric problems related to Easter day. Pacioli's Suma contained, apart from the problems of the four operations (for check of multiplication he recommended "the method of 7"), problems on coins' conversion, progressions, interests, undetermined analysis, equalization as well as assignation of the perfect number. It also contained geometry problems. It should be noted, that for the calculation of the side of a regular 9- gon he calculated the 1/4 of the sum of the sides of an equal sided triangle and of a regular hexagon. In Codex 65 the side of a regular 9- gon equals to one half of the sum of the sides of a regular octagon and a regular decagon [34].

It is certain that many Latin scholars who knew ancient Greek read Greek manuscripts and were influenced by them. At the Mantua School in Italy, (c. 1430 A. D.) Vittorino da Feltre taught mathematics using Greek manuscripts. After his death his library came into the possession of Jacopo Cassiano who continued to teach mathematics at the same school [38].

Thus in this case in order to reach certain conclusions, a comparison between the contents of those Italian works and that of Codex 65 is required. However, that is outside the framework of this study.

## Roots of real numbers

The chapter on the roots belongs to material, which is clearly algebraic eventhough the solutions are also given within the well-known form of recipes. However they are thorougly distinguished by what to being said previously because of their subject and classification by the author in the second book. In the first part of his 2$^{nd}$ book (chapter 117, f. 62v) the author relates, that there are problems, which cannot be solved using the methods in the first book. He also writes that he prefers to give "alternative and dissimilar handling" with which the following problems are solved.

The material of algebra includes the roots of real numbers, equations up to fourth level, and the system of equation up to second level.

According to what is held so far, all the algebraic knowledge [3] acquired by Europeans during the 15$^{th}$ century originated on the one hand from the ''Algebra'' of Al Khwarizmi (11$^{th}$ century) and on the other hand, from the work ''Liber Abbaci'' by Fibonacci (13$^{th}$ century) [39]

Al Khwarizmi wrote two books on arithmetic and algebra [4], the Latin translations of which containes roots, equations, etc. However, the *Algebra* of Al Khowarizmi gives the impression that the author was influenced by more ancient sources than the Greek and Hindu ones, such us the mathematics of Babylonia [70]. Because in the Arabic algebra there is not any "indeterminate analysis" which the Hindu held particularly in great esteem, it may be construed that these must have been sources from Mesopotamia [5]. The Arabs, however, had received all their knowledge of Algebra and astronomy from the Greeks [52]. According to Van der Waerden [71], the fact that the Babylonians, the Hindu, the Greeks and the Chinese, used, apart from the Pythagorean rule, also the Pythagorean triads, leads to the conclusion that the origin of the theories should have been common for the four civilizations, and that his common origin must be dated prior to the discovery of the writing.

Apart from Al Khowarizmi, the following Hindu wrote in relation to roots: Sridhara ($11^{th}$ century), Brahmagupta ($7^{th}$ century) Bhascara ($12^{th}$ century) [35], whilst on matters of algebra Aryabhata ($5^{th}$ century) and Mahavira ($9^{th}$ century).

In this unit included are the "Rules of multiplication", meaning the multiplication tables of natural numbers from 1 to 1000 (chapter 127, f. 64v), as well as the corresponding charts of their roots. The author Rabdas included multiplication tables in his work in 1341, from whom, possibly, the Italians got the idea. In the codex 65 however, the author gives tables of root calculations for some numbers from 1 to 1000, (chapter 239, f. 117v). Although the Byzantines in general did not use such tables (not even the table on calculations of the squares), it appears that some teachers, such as the one of Codex 65, included them in their material, maybe for educational reasons so to give to their students the possibility to find immediately the true root of a natural number. Of course there are tables with square numbers in Boethios' arithmetic ($5^{th}$- $6^{th}$ century) in Italy. The fact, however that Rabdas had not included such tables in his work indicated that maybe in those times, the Greeks did not use tables of numbers' squares and that the simple multiplication tables were sufficient. In a manuscript in Prague in 1300, was found the work "Crafte of Nombrynge", in which included are square numbers. In relation to these Jordanus Nemorarius (1225) Rolland (1424) Al-Kashi (1430) also wrote [53].

The starting point in algebra for the author of codex 65 is the chapter on calculations of roots of real numbers. Other Byzantine scholars such as Isaac Argyros [28] (1310–1371) and Maximus Planudes (circa 1300) had also got involved with roots.

In accordance with the methods of calculation of the square and the cubic root in Codex 65 [12], it appears that the root of 30 is equal to 5 5/11 (chapter 123, f. 64v). The preferred method appears to be the same as that of Omar Khayyam (1048–1131) and is the following: If $N^{1/\eta} = \chi$ with $N = \alpha^{\eta} + \tau < (\alpha + 1)^{\eta} - a^{\eta}$ then the following will be valid:

$$\{(\alpha^{\eta} + \tau)\}^{1/\eta} = \alpha + \tau / ([\alpha+1]^{\eta} - \alpha^{\eta}).$$

If we put $\eta = 2$ and $N = 30$, then according to this form we get: $N = \alpha^{\eta} + \tau$, e.g. $30 = 5^2 + 5$, with $5 < 6^2 - 5^2$ or $5 < 11$ and $\sqrt{30} = \sqrt{(5^2 + 5)} = 5 + 5/(6^2 - 5^2) = 5 + 5/11$.

If the calculation of the root of 30 is done with the method used by Planudes, which is based on the formula of the Hero of Alexandria [23], we will have: $\sqrt{N} = \sqrt{(a^2 + \tau)} = \alpha + \tau/(2\alpha)$. E.g. $\sqrt{30} = \sqrt{(5^2 + 5)} = 5 + 5/10$ and not $5 + 5/11$.

From a comparison between the method of the author of Codex 65 and that of Rabdas, at first glance it appears that the latter used Hero's formula, and that also he further considered that $\alpha = \alpha + \tau/(2\alpha)$. If $\alpha$ had been the higher approximation of the root, then the $\alpha_1 = N/\alpha$ was the less approximation, and the rate $(1/2)(\alpha + \alpha_1)$ was considered from the author of the codex 65 as the better of these [24].

According to the above, we would have the following:

$$\sqrt{30} = (1/2)\{5 + 5/10 + 30/(5 + 5/10)\} = (1/2)\{55/10 + 30/(55/10)\}$$
$$= (1/2)(11/2 + 60/11) = (1/2)\{(121 + 120)/22\} = 241/44 = 5 \cdot 21/44.$$

We observe that, when in the codex 65 is given approximately the root of 30, then the number $5 \cdot 21/44$ is found as the second approximation of this root (chapter 123, f. 64v, 65r), which agrees with the second approximation which is found by Rabdas, although their values for the first approximation do not agree; in the codex 65 is found the number $5 \cdot 5/11$ while Rabdas gives $5 \cdot 5/19$.

In TractMathVindGr however, there is a reference to another method of calculation of the square root (chapter 240, f. 124r), according to which the square root of 98 is found to be equal to $9 \cdot 18/20$. This method does not agree with any other well-known method of that time, because if we apply the formulas, e.g. that of Khayyam and Planudes for the square root of 98, we may equally find the values $9{\cdot}19/21$ and $9{\cdot}18/20$.

We should also emphasize, that in TractMathVindGr, there are no mathematical formulas but only instructions for the calculation of the square root. Despite the fact that, since 275 A.D. Diophantus had already introduced his own symbols [54], they were not used, maybe because the description of the formulas, in those times, was easier to be understood.

Barlaam Calabros knew the formulas which we referred to. According to him, the procedure of approximation could be continued [7] by applying the formula $\chi_{\eta+1} = (\chi_\eta + N/\chi_\eta)/2$, where $\eta = 0,1,2,3,\ldots$

But in TractMathVindGr, there is a described method which may lead to successive approximations.

We conclude then, that the author, as far as the method of extraction of the square root is concerned, had possibly accepted influences, especially from Omar Khayyam; and this is why his method was different than that of his counterparts.

The methods of calculating a square root, which we referred to above, seem to have been abandoned within the years, and finally in the year 1494 Luca Pacioli gives a method, similar to the this one which was taught at schools of secondary education some years ago. Later, in 1546, Cataneo reaches more this method [55], which reminds the art of division and raises particular difficulties, for the students, in memorizing.

It would be useful to mention, that the old method may be superior to the modern one concerning the question of easily in memorizing, just because the operations are clearly less. Furthermore, an important disadvantage of the modern method is the fact that the students cannot justify the particular procedure, with the result that very few of them could remember even its beginning.

As far as the cubic root, all things were entirely different. There was no standard method of calculation and the procedure of finding its solution was considered to be particularly difficult. The author of codex 65 writes that he is not familiar with particular method (chapter 125, f. 66r). In 1599 Buteo succeeded in calculating only the first digit number of a cubic root. A hundred years later, Lagny believed that more time was required in order to find out the cubic root of a number [56]. Nonetheless they had at their disposal the general formulas of calculation roots. Therefore it remains inexplicable why they didn't apply the general forms for the cubic root as well like they used to do for the square root. The most surprising thing in codex 65 is mentioned that there was no method available, in fact, further below, the calculations are done with the same ease the square roots have been calculated, and the methodology, which is followed, is of the same form as that of the square root:

Of course, this question troubled other scientists as well, during that time, such as Mahavira (9th century) [57] and Fibonacci [66]. Omar Khayyam gave a general formula of finding the root of n-th order, as it was mentioned before.

But, according to Hero [22], if $\alpha^3 < N < (\alpha+1)^3$, then $N - \alpha^3 = \beta$ and $(\alpha + 1)^3 - N = \gamma$, therefore $N^{1/3} \simeq \alpha + \{(\alpha + 1)\beta\}/\{(\alpha + 1)\beta + \alpha\gamma\}$ [72].

If we apply the above formula for $N = 30$ we would have $3 < \sqrt[3]{30} < 4$, because $3^3 = 27$ and $4^3 = 64$, therefore $64 - 30 = 34$ and $30 - 27 = 3$, meaning

$$\sqrt[3]{30} = 3 + (4{\cdot}3)/(4{\cdot}3 + 3{\cdot}34) = 3 + 12/(12 + 102) = 3{\cdot}6/57.$$

Now according to Fibonacci we have $\sqrt[3]{N} = \sqrt[3]{(\alpha^3 + \tau)}$, therefore $\sqrt[3]{N} = \alpha + \tau/\{(\alpha + 1)^3 - \alpha^3\}$, therefore $\sqrt[3]{30} = \sqrt[3]{(3^3 + 3)}$, and as a result $\sqrt[3]{30} = 3 + 3/(4^3 \cdot 3^3) = 3{\cdot}3/37$.

According to Omar Khayyam, we would have exactly the same result as in the method of Fibonacci and Hero.

In TractMathVindGr the cubic root of 30 is found equal to $3{\cdot}3/37$.

In this article I have presented some few results of my study on the mathematical content of the published part (f. 11r–126r) of the Codex Vind. Phil. gr. 65 (Tractatus Mathematicus Vindobonensis Graecus or TractMathVindGr). This 15th century (1436) Byzantine MS includes the solution of problems of practical arithmetic, algebra and geometry, the roots of which can be traced back to antiquity and their comparison with modern mathematical solutions reveals—apart from some differences—many identities and similarities showing the unbroken continuity of mathematical tradition through the centuries. Moreover, my research has revealed so far some important results according to which we are probably in the position to give to the TractMathVindGr the title of the Byzantine encyclopaedia of Mathematics.

**References**

[1] Adel Anbouba, L' Argèbre Al-Badi d' Al-Karagi, Pub. Univ. Libanaise, Beyrouth 1964, p. 34.
[2] These sums are found in the work of Avicenna, Danesh-Nama, which was written in Persian in the 11th c. A.D. See Avicenne, Le livre de Science, trad. Moh. Achena et Th. Massé, Les belles lettres, 1986, p. 195.
[3] The major creator of algebra was considered to have been the Hindu Baskara (1114-1185 A.D.) who was influenced by old Hindu sources. See Boyer- Merzbach, Hist. Math., p. 248.
[4] Boyer- Merzbach, Hist. Math. p. 255, V.d. Waerden, Awakening, p. 328.
[5] Boyer- Merzbach, Hist. Math, p. 257, 258.
[6] C. B. Boyer – Uta. C. Merzbach, "History of Mathematics", ed. A. Pneumatikou, Athens, 1997, p. 277.
[7] The full text for the discovery of the square root per Barlaam von Seminara Logistike, P. Carelos, Academy of Athens, 1996, p. 114.
[8] Maria D. Chalkou, The mathematical content of the codex Vindobonensis phil. graecus 65, ed. Byzantine Research Center of Aristotelean University of Thessaloniki, Thessaloniki 2006.
[9] Maria D. Chalkou. Mathematical education and its terminology in the Byzantine period, according to the Codex Vindobonensis phil. gr. 65, "Eoa kai Esperia", vol. V, ed. Etaireia Ereunas ton scheseon tou messaionikou kai neou ellinismou me ti dyssi, Athens 2001-2003, p. 51- 62.
[10] Maria Chalkou, Problems in multiplication, division, analogies and progress according to Codex Vindobonensis phil. Gr. of the 15th Century. Second meeting of Byzantine Scholars of Greece and Cyprus, Athens 1999, p. 172.
[11] Maria D. Chalkou, The Mathematical content of the Codex Vindobonensis Phil. Gr. 65, ed. Research Byzantine Center of Aristotelean University of Thessaloniki, Thessaloniki 2006, chapter 71. [12] Maria D. Chalkou, About roots, 3rd Meeting of the Byzantinologists of Greece and Cyprus, Rethymno 2000, p. 172.
[12] C. N. Constantinides, Higher Education in Byzantium in the thirteenth and early fourteenth centuries (1204-1310), Cyprus Research Center, Nicosia, Cyprus 1982, p. 73, 157.
[13] Diophantos , Arithmetica, ancient text and translation by E. Stamatis, OEDB, Athens, 1963, p. 45
[14] Euclides, Geometry, ed. E. Stamatis, OESB, Athens, 1958, vol. II, p. 24-67. Euclides, Elementa, ed. 1. L.Heiberg, Teubner, Lipsiae, 1884, vol. II, p.271 Euclid: The thirteen books of the Elements, translated with introduction by Th.Heath, vol. II, Dover, 1956, p. 138-186.
[15] According to Manuel Glyzonios: "Draft Book for all uses, containing Practical Arithmetic, or rather, Calculation", Venice, 1569. See: N. Georgakopoulos Education in Arcadia under Turkish occupation, ed. Phylla, Tripoli, 2000, p. 132-135.
[16] N. Georgakopoulos, Education in Arcadia under Turkish Rule, ed. Phylla, Tripoli, 2000, p. 132, 135.
[17] Nicomachi Geraseni, Pythagorei Introductionis arithmeticae libri II, ed. Hoche, Teubner, Lipsiae, 1866, p. 148-149.
[18] M. Giozzi, Cardano, Girolamo, DSB, vol. III, p. 64-67
[19] Planudes used checking methods of the operations of multiplication and division in which he used the division by 9. See Th. Heath, A History of Greek Mathematics, Oxford UP., vol. I (1921), p. 116.
[20] Heath, Hist. Gr. Math, vol. II, p. 441. E. Stamatis, Diophantus Arith. p. 212, book IV, prob. 31. This problem is solved using the method of "false assumption".
[21] Heath, Hist. Gr. Math., Vol. II, p. 341. [62] Greek translation of: History of the Byzantine Empire, (v. II, ch. XXVIII: K. Vogel, "The Byzantine Science"), Univ. of Wisconsin Press, Cambridge, 1958, p. 815.
[22] Th. Heath, Hist. Gr. Math., Vol. II, p. 547- 548. Heron usually stopped to the finding of the first approach. See A.G. Drachmann- M. S. Mahoney, "Hero of Alexandria", DSB, vol. VI, p. 310- 315, 314.
[23] Th. Heath, Hist. Gr. Math., p. 553.

[24] H. Hunger, Die hochspradilihe Literatur der Byzantiner (Greek translation), vol. I-III ed. MIET, Athens, 1994, vol III, p. 42, 49.
[25] H. Hunger- K. Vogel, Ein Byzantinisches Rechenbuch des 15 Jahrhunderts. 100 Aufgaben aus dem Codex Vindobonensis Phil. Gr. 65, H. Bohlaus, Koln Komm. d. Österr, Acad. d. Wissenschaften in Wien, 1963, Moirasmos tzakismaton, p. 78.
[26] Problems which are solved by the "rule of three". See Hunger- Vogel, Byz. Rechenb., p.16.
[27] Hunger, Byz. Literature, Vol. III, p. 59.
[28] The book "Commercial Mathematics" of Pacioli was written circa 1470- 1481 but was never published. The "Suma" was published in 1494 and its 600 pages contained tables of coins, weights and measures relating to the Italian cities. It was a work of great and was taught even in the $16^{th}$ century. On this book Nicolo Tartaglia based his work "General trattato de numeri et misure" (1566- 1560). See S.A. Jayawardene, "Luca Pacioli", DSB (Dictionary of Scientific Biography) ed. Ch. Coulston Gillespie Ch. Scribners sons. Vol. I- XVI, N. York 1970- 1980), vol. X, p. 269- 272.
[29] Jaywaerdene, Pacioli, p. 270- 1.
[30] Jaywaerdene, Pacioli, p. 270- 1.
[31] E. Knobloch, Mathematiques et Philosophie de l' Antiquité à l' âge Classique, CNRS, Paris, p.218-221.
[32] P. Lemerle, The First Byzantine Humanism (Greek translation) - ed. MIET. Athens 1985, p. 33-35.
[33] G. Loria, History of Mathematics, ed. Papazissi, Athens, 1971, vol. II, p. 350, 360- 374.
[34] The last two knew the form $\sqrt{a}+\sqrt{b}=\sqrt{(a+b+2\sqrt{ab})}$. See Loria, Hist. Math, vol. II, p. 239.
[35] N. Nikolaou, Theoretical Arithmetic for High Schools, ed. D. Tsaka- St. Delagrammatika, Athens, 1954, p. 39, 40.
[36] P. L. Rose, "The Italian Renaissance of Mathematics", Librarie Droz, Geneva, 1975, p.143.
[37] Rose, Ital. Ren. Math., p. 17.
[38] The mathematical works of Fibonacci and Jordanus Nemorarius provided worthy service in the rebirth of Mathematics, of the same family as those of the Greek manuscripts. See Rose, Ital.Ren. Math. p. 75. Further in the work "Liber Abbaci", by Fibonacci there are methods of extracting square and cubic root. See Smith, Hist. Math, vol. I, p. 216. Particularly the methodology of finding a cubic root, which is not included in all books on algebra, leads to the question, as to how much of the work writer could have been influenced by Codex 65.
[39] This Italian arithmetic includes other methods of multiplication equation. See D. E. Smith, History of Mathematics, I- II, Dover, New York, 1958, p. 81, 110.
[40] D. E. Smith, "History of Mathematics", vol. I- II, Dover, New York, 1958, p. 112- 114.
[41] As above, p. 136.
[42] As above, p. 137.
[43] Smith. Hist. Math. vol.II p.151, 154
[44] Smith, Hist. Math., vol. II, p. 155.
[45] Smith, Hist. Math., vol. II, p. 483, 486, vol. I, p. 274.
[46] E.g. the solution of the equation $\chi + \chi/7 = 19$ is the same as that of problem No 1 of Papyrus of Ahmim, (c. 1550 B.C.). See Smith, Hist. Math., vol. II, p. 582.
[47] As above, p. 588.
[48] Smith, Hist. Math., vol. I, p. 216. [47] E. g. the solution of the equation $\chi + \chi/7 = 19$ is the same as that of problem No 1 of Papyrus of Ahmim, (c. 1550 B.C.). See Smith, Hist. Math., vol. II, p. 582.
[49] Smith, Hist. Math. Vol. II, p. 532.
[50] Smith, Hist. Math. Vol. II, p. 546. [67] B.L. Van der Waerden, "Science awakening", Greek translation. ed. Univ. of Crete Herakleio 2000, p. 58.
[51] Smith, Hist. Math, vol. I, p. 289.
[52] Smith. Hist. Math, vol. II, p. 124, 126.
[53] Smith, Hist. Math., Vol. II., p. 382. Diophantus, Arith., p. 98.
[54] Smith, Hist. Math., vol. II, p. 146.
[55] As above, p. 146.
[56] As above, vol. I, p. 162.
[57] E. Stamatis, Greek Mathematics, ed. Etaireia ton filon tou laou, Athens 1979, p. 69.
[58] As above, vol. I, p. 298.
[59] E. Stamatis, Greek Mathematics, Athens, [2]1979, p. 104. As above Nicomachi Geraseni, Introductionis arithmeticae libri II, ed. Hoche, Teubner, Lipsiae, 1866, p. 150.
[60] J. H. Vincent, A la géometrie pratique des Grecs. Extrait des notices des Manuscrits, vol. XIX, pt. 2, Imp. Impériale, Paris 1858, p. 204. G. Pachymeris, de Michaele et Andronico Palaeologis bonnae impensis, ed.Weberi 1835 (2), p. 233.
[61] Greek translation of: History of the Byzantine Empire, (v. II, ch. XXVIII: K. Vogel, "The Byzantine Science"), Univ. of Wisconsin Press, Cambridge, 1958, p. 815.

[62] K. Vogel (trans. K. N. Sidiropoulos), "Engrammatos logismos kai indika psifia sto Byzantio", NEFSIS 5, (Autumn-Winter 1996) 80. The original article is in: Des XI. Internationalen Byzantinistenkongresses 1958, appl. Fr. Dölger and H. G. Beck, Munich, C. H. Beck'sche Verlagsbuchhandlung 1960, p. 660-664.
[63] The second part of the work Liber Abbaci by Fibonacci, contains problems in the nature of enigmas which are solved with linear equations. These are frequently solved by using the method of false assumption. See K. Vogel, "Leonardo Fibonacci", DSB, vol. IV, p. 604-613, p. 606.
[64] Vogel, Fibonacci, DSB vol. IV, p. 604- 613, 606.
[65] Vogel, Fibonacci, p. 607.
[66] B.L. Van der Waerden, "Science awakening", Greek translation. ed. Univ. of Crete Herakleio 2000, p. 58.
[67] According to Freudenthal, the Hindu of 200- 600 A.D. knew Greek astrology, the sixtieth system and the zero. In their Brahman digits they added the Greek "o" (the initial of the word "οὐδέν") and they adopted the Greek-Babylonian order. See B. L. Van der Waerden, "Science awakening", Univ. of Crete, Herakleio 2000, p. 55.
[68] Papyrus Rhind: Problem No 26. "One number and its fourth part gives together 15" To find the number, the author accepts as a solution the number 4, where 4+1=5 and not 15. As 15 is the triplication of 5, thus the number sought will be the triplication of 4, e.g. 12. See V. D. Waerden, "Awakening of science", p. 19.
[69] B. L. van der Waerden, A History of Algebra, Springer Verlag 1985, p. 13- 15.
[70] V.d. Waerden, Geom. Alg. Anc. Civ. p. 15.
[71] B. L. Van der Waerden, Geometry and Algebra in Ancient Civilisations. Springer, Berlin 1983, p. 187.

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