Преглед НЦД 9 (2006), 63–70

Maria D. Chalkou
(Department of Mathematics,
National University of Athens)

THE PYTHAGOREAN RULE AND ITS APPLICATION, ACCORDING TO A GREEK MANUSCRIPT OF THE 15TH CENTURY IN WIEN

Abstract: The Code 65 (Codex Vindobonensis phil. graecus 65) is made of paper and dates from the 15th century. In this Code (chapters 167–184), included are problems which are solved mainly by the use of the Pythagorean rule or "the rule of skadra" (κανών της σκάδρας), as this was called by the author of the Code. This article contains some chosen problems of chapters 167–184 and their mathematical comments. We tried an interpretative approach of the methods of the Code's author, as well as comparison of these with the corresponding ones of today, which are taught in the secondary education.The methodologies of finding answers even if in some cases they are not probably used by the mathematicians of the secondary education, as we ascertain from the examples which follow, adequate common factors with those used today in corresponding problems.

Key words: mathematical Byzantine manuscript, logisticae and geodesia in Byzantium.

The Code means that the pages (or the volumes) are not joined into a roll, but were simply placed between wooden frames or frames of some other material. It is considered to be a pioneering form of today's book. It appeared in the 2nd century and due to its ease in use it replaced the cylinder. At about the same time, the parchment (usually the hide of a cow of sheep) as writing material, replaced the papyrus. The Code consists of joined volumes and each volume of one (alternating) page number, folded in half.

The author and the origin of the Code 65 are unknown. Augerius von Busbeck got the Code when he was Ambassador to Emperor Ferdinand I in the courtyard of Sultan Souleiman II (1555–1562). The pages 126v–140r contain a book of Arithmetic (Arithmetic includes logisticae and geodesia) with solved problems, which was published by H. Hunger and K. Vogel in 1963. The larger part of the Code (f. 11r–126r) contains an anonymous arithmetic book with 240 chapters of which, the preface and the first two chapters were published by J. L. Heiberg in 1899. The eleventh unit of the unpublished section of the Code (chapters 167–184), includes problems which are solved mainly by the use of the Pythagorean rule or "the rule of skadra" (κανών της σκάδρας). In this article we have chosen some problems of chapters 167–184 and we have presented their mathematical comments. We have primarily tried an interpretative approach of the methods of the Code's author. Particularly interesting is the comparison of the teaching methods of now and then, while the codex 65 according to our assessments was also aiming at the teaching of students at various levels of education.

Chapter 167. Finding the diameter of a circle from its perimeter. The author of the Code, takes the perimeter to be equal to 22 and supposes that the ratio of the perimeter to the diameter is always equal to $3^1/_7 = 22/7$. As a result, the required diameter will be equal to 22/(22/7) = 7. Clearly, for him the $3^1/_7$, which is an approximation of π = 3,14159…, represents π.

Chapter 168. Finding the perimeter of a circle from its diameter. He uses the same ratio $P/d = 3^1/_7$, where P symbolizes the perimeter and d the diameter of a circle. In the manuscript of course, there are no symbols, but only the corresponding words of the perimeter and the diameter.

Chapter 169. Finding the diameter and circle's perimeter, from the chord length and from the distance of the middle of it from the peripheral. According to the code we have: MZ = 2, MK = 1, MN = 4, KL = 3 (fig. 1). Today we would call r the radius ON of the circle, where according to the Pythagorean rule we would have: $r^2 = 4^2 + (r - 2)^2$, from which r = 5, and d = 10.

The author uses the following procedure 4.4 = 16, 3.3 = 9, 1.1 = 1, 9+1 =10, 16 –10 = 6, 2.1= 2, 6/2 = 3, 3.3 = 9, 9+16 =25, the root of 25 is equal to 5 and 5.2 = 10 = d the diameter of the circle.

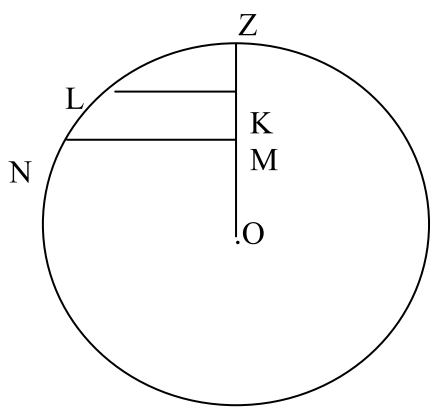

Fig. 1

According to the author of the code: $KL^2 + MN^2 = r^2$, but since $OM^2 + MN^2 = r^2$, it must happen that OM = KL. But we have:

$$OM^2 = ON^2 - NM^2 = r^2 - 4^2 = OZ^2 - 4^2 = (2+ OM)^2 - 4^2,$$

thus $OM^2 = 4 + OM^2 + 4OM - 16$, or $4OM = 12$, so OM = 3 = KL, which is real as he considers that KL = 3

Chapter 170. Finding the side of a registered square if the circle's diameter is equal to 6. The diagonal of the square is considered equal to the diameter of the circle (Fig. 2) and he finds the side of the square using the Pythagorean rule. It appears to be a simple application of the theory, which we also use today.

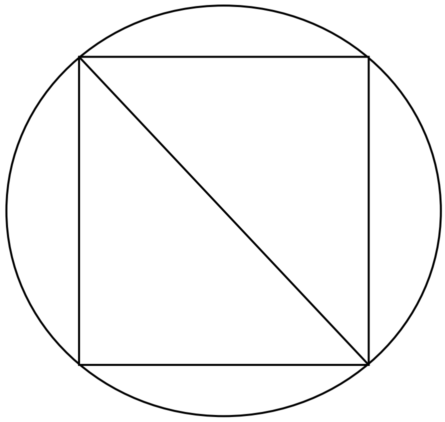

Fig. 2

Chapter 171. The author of the code wants to construct a square DEGF, whose side is equal to the section FG of the circle's diameter AB = 12 (Fig. 3)

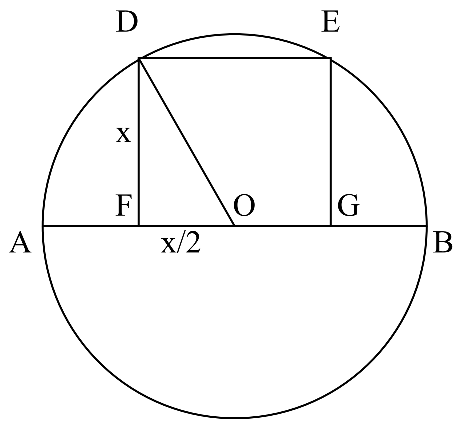

Fig. 3

The author describes only the last step as he calculates directly x as the square root of $AB^2/5$. Today, we would consider the side of the square equal to x and if we apply the Pythagorean rule we would have: $OD^2 = FD^2 + FO^2$, namely $6^2 = x^2 + (x/2)^2$, or $x^2 = 144/5$, or $x = \sqrt{(144/5)}$.

Chapter 172. In this particular problem he asks to draw a circle within a square having side equal to 7. He considers the diameter of the circle to be equal to the side of the square and multiplies the 7 by $3^1/_7$, as he normally does, considering π as an approach equal to $3^1/_7$.

Chapter 173. He asks for a circle to be constructed within a rhombus having sides equal to 7. He also considers the small diagonal of the rhombus to be equal to 7.

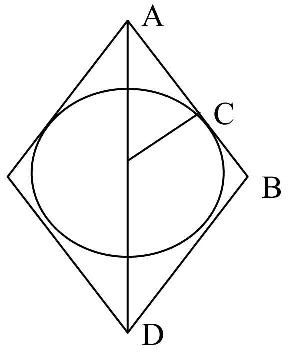

Fig. 4

The author finds OA using the Pythagorean rule. He writes:

$$7 \cdot 7 = 49,\ (7/2) \cdot (7/2) = 12^{1}/_{4},\ \ 49 - 12^{1}/_{4} = 36^{3}/_{4},$$
$$\sqrt{(36^{3}/_{4})} = 6^{1}/_{16},\ \ (6^{1}/_{16}) \cdot 2 = 12^{2}/_{16} = AD \quad \text{(Fig. 4)},$$

and he says that the diameter of the circle is equal to $6^{1}/_{16}$. This is clearly valid, because $OA = (7\sqrt{3})/2 = 2OC$ ($\hat{A} = 60^{o}$, and $O\hat{A}C = 30^{o}$)

Chapter 174. Calculation of the side of an equilateral triangle drawn in a circle of a diameter 12. The author calculates that the height AE = ($^{3}/_{4}$) 12= 9 (Fig. 5), and writes: 81 + 81/3 = 108. Then he finds the root of 108, which he considers to be equal to $10^{7}/_{8}$, and considers this to be sought for side of the equilateral triangle.

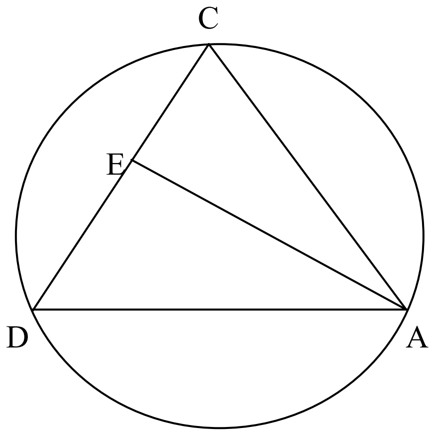

Fig. 5

Today, if r symbolizes the circle's radius and O the circle's center we would write:

$$AD^2 = AE^2 + ED^2 = (r + r/2)^2 + ED^2 = (3r/2)^2 + ED^2$$
$$= 9r^2/4 + OD^2 - OE^2 = 9r^2/4 + r^2 - r^2/4 = 3r^2 = 3 \cdot 36 = 108,$$

thus AD is equal to $\sqrt{108}$. The author clearly argues that $AD^2 = AE^2 + (1/3)\ AE^2$, but we can write $AE^2 + (1/3)\ AE^2 = AE^2 + (AE\sqrt{3}/3)^2 = AE^2 + \{[(a\sqrt{3}/2)\sqrt{3}]/3\}^2$, because, if we denoteby a the side of the triangle ACD, then $AE = a\sqrt{3}/2$. So

$$AE^2 + \{[(a\sqrt{3}/2)\ \sqrt{3}]/3\}^2 = AE^2 + (a/2)^2 = AE^2 + ED^2,$$

Namely $AD^2 = AE^2 + ED^2$, and this is true according to the Pythagorean rule.

Chapter 176. Calculation of the circle diameter inscribed in an equilateral triangle whose side is equal to 4.

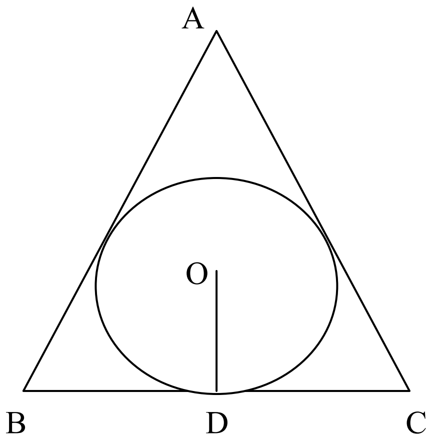

Fig. 6

The author writes that the square of the triangle's side is equal to 16, since the side is equal to 4. Then he writes: 16 – 16/4 = 12 and that the height AD (Fig. 6) of the equilateral triangle is equal $\sqrt{12}$, which is equal to $3^{8}/_{17}$. The diameter of the circle is equal to 2/3 of $3^{8}/_{17}$ or equal to $2^{16}/_{51}$. Then the perimeter is calculated by multiplying the diameter by $3^{1}/_{7}$.

Today, we would calculate the section OD $=1/3 \cdot (4\sqrt{3}/2) = 2\sqrt{3}/3$, where the perimeter would have been equal to $(4\sqrt{3}/3)\pi$.

Chapter 177. Finding the side of the square ZHED when the side of the inscribed equilateral triangle ACB is equal to 10 (Fig. 7)

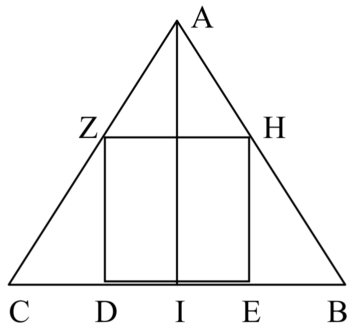

Fig. 7

The procedure presented in the manuscript, is the following:

$$10 \cdot 10 = 100,\ 100 \cdot 3/4 = 75,\ 75 \cdot 16 = 1200,\ 75 \cdot 12 = 900,\ \sqrt{1200} = 34^{12}/_{19},$$
$$\sqrt{900} = 30,\ 34^{12}/_{19} - 30 = 4^{12}/_{19} = x,\quad \text{where x is the side of the square.}$$

We would take the side of the square equal to x and since the triangles CZD and CAI are similar, we would write: DZ/AI = CD/CI, namely $x/(10\sqrt{3}/2) = (5 - x/2)/5$, wherefrom $x = 2\sqrt{3}(10 - 5\sqrt{3})$. We understand therefore that in the manuscript the difference $\sqrt{(75 \cdot 16)} - \sqrt{(75 \cdot 12)}$ is equal to x because it is equal to

$$\sqrt{3}\sqrt{(25 \cdot 16)} - \sqrt{3^2}\sqrt{(25 \cdot 4)} = 2\sqrt{3} \cdot 10 - 2\sqrt{3^2}\sqrt{25} = 2\sqrt{3}(10 - 5\sqrt{3})$$

Chapter 179. Calculation of the diameter and perimeter of the circle inscribed in a rectangular triangle having sides of 3, 4 and 5.

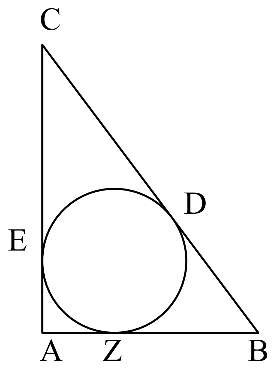

Fig. 8

The author, having analytically explained the method of calculation of each side of the rectangular triangle from the two others, applies three times the Pythagorean rule, adding the two vertical sides and arrives at 3 + 4 = 7. Further, he subtracts 5 from 7 and gets 2. He deduces therefore that the diameter of the circle is equal to 2.

Today, we would write $x = AE = AZ$, $j = DB = BZ$, and $z = CD = CE$ (Fig. 8), and would have $2x + 2j + 2z = 2\tau$, where the τ denotes the half perimeter of the rectangular triangle. So $x = \tau - (j + z) = (3 + 4 + 5)/25 = (3 + 4)/2 - 5/2 = 7/2 - 5/2 = 1$, where the radius of the circle is equal to 1, the diameter equal to 2 and the perimeter equal to $2 \cdot (3^1/7)$.

Chapter 180. Calculation of the side of the square inscribed in a rectangular triangle with sides 3, 4 and 5 as depicted by Fig. 9.

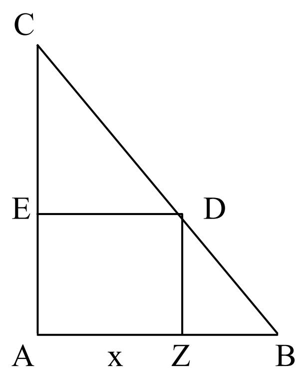

Fig. 9

Let AB = 3, AC = 4, BC = 5 and ED = x. Since the triangles CED and CAB are similar, we have ED/AB = CE/AC, or $x/3 = (4 - x)/4$, wherefrom $4x = 12 - 3x$, and $x = 12/7$.

In the manuscript a fundamental rule of calculation is given, i.e., $bc/(b + c) = x$, where b and c are orthogonal sides of the triangle. In our case $4 \cdot 3 / (4 + 3) = 12/7$.

The author is involved exclusively in the Geometry of Eyclides [6] to which he applies exercises of a clearly practical content, without however, the absence of the theoretical questions. He defines the figure π as the ratio of the perimeter of the given peripheral of a circle to its diameter and considers it to be equal to 22/7, or $3^1/_7$. This approach, without of course it being satisfactory, was accepted by Archimedes himself, since he would use it in practical measurements [3], [4]. We can therefore assume, that the reason why the author, even in matters of theoretical content, uses the ratio $3^1/_7$, is because these questions make up a basic tutorial in the problems which were studied, which were practical problems of everyday life.

The approachable ratios of π presented an interesting variety so much in the 15th century and so much in older times. As an indication we state that in the 10th century, the Hindu Sridhara put $\pi = \sqrt{10}$ [7] and in the 13th century the Chinese Ch in Kiushao gave a ratio for π of $3^1/_7$ [8]. In Egypt in olden times, the value 3,1605 [5], was used. Michael Psellos considered that π was equal to $\sqrt{8}$ =2,8284271 [1] . After Archimedes, the ratio of $3^1/_7$ was adopted by Heron of Alexandria [2] and in the West by Fibonacci (13th cent. A.D.) [10], Dominicus Parsiensis (1378 A.D.), Albert the Saxon (1365 A.D.) and Nicholas of Couza (1450 A.D.) [9].

Chapter 182. Calculation of the "height" AD, of a triangle ABC, when AB = 16, AC = 12, and BC = 14 (Fig. 10).

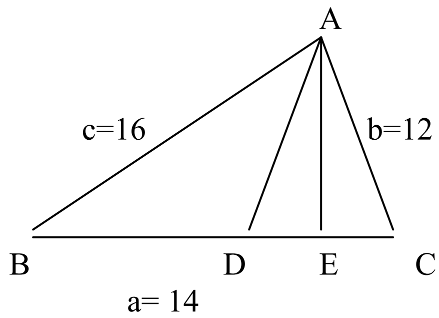

Fig. 10

The procedure in the manuscript is the following: 16/2= 8, BD= 8, $14^2 = 196$, 196 + 8 = 204, $2 \cdot 2 = 4$ (2 is the difference between the sides), 204 + 4= 208, $16 \cdot 16 = 256$, 256/4 = 64, 208 – 64 = 144, and AD is equal to $\sqrt{144} = 12$. According to this method, if AB = c, AC= b, and BC= a, the following will apply: $AD^2 = a^2 + c/2 + 2(c - a) - c^2/4$. But $a = c - 2$, so $AD^2 = (3c^2 - 14c + 32)/4 = (3 \cdot 16^2 - 14 \cdot 16 + 32)/4 = 576/4 = 144$, so AD = 12.

From $AD^2 = bc[1 - a^2/(b+c)^2]$, which today is used when we have to calculate the line AD which separates the angle A of the triangle ABC into two equal angles, if $a = c - 2$ and $b = c - 4$ we will have: $AD^2 = (3c^2 - 12c)/4$, resulting in $AD^2 = (3 \cdot 16^2 - 12 \cdot 16)/4 = (3 \cdot 256 - 192)/4 = 144$ and AD= 12.

The two methods produce the same result. Further, observe that $AD^2 = (3c^2 - 14 \cdot c + 32)/4$, to which we arrive by applying the author's method, takes the form:
$AD^2 = (3c^2 - 12c - 2c + 2 \cdot 16)/4$, as a result $AD^2 = (3c^2 - 12c)/4$. (because c = 16, then $-2c+32 = 0$).

We notice that the author of the codex 65 writes "vertical", but he means the line AD which separates the angle A of the triangle ABC into two equal angles and not AE which is vertical to the side BC.

From the comparison of the anonymous author's methods with the corresponding ones of today, which are taught in the secondary education, we realize that, regarding the problems solved by the use of the Pythagorean rule, he fundamentally uses the same types that we use today, which however, appear in a different way. We also suppose that he is aware of the relative theory even if he doesn't explain the origin of his methodologies.

References

[1] Th. Heath, *A History of Greek Mathematics*, Dover, New York, 1956, vol.II, p.545.

[2] Heronis Alexandrini, *Stereometrica et de mensuris*, ed. J. Heiberg, Teubner Stutgard, 1976, vol.V, p.20, 190. See also: H. Vincent, *À la Géomètrie Pratique des Grecs*, vol.XIX, p.216.

[3] From the work of Archimedes Circle Measurement the following corresponding relations are arrived at for π: $\pi<3\ 1/7$, $\pi>3\ 10/71$. See W.R. Knorr *Archimedes and the measurement of the circle,* A new interpretation, AHES, 15. N.2 (1976).

[4] *Archimedes all the works*, ed. E. Stamati, TEE, Athens, 1974, part B, vol. I, p. 454.

[5] J. Lefort, R. Bondoux, J-Cl. Cheynet, J.-P. Grélois, V. Kravari, *Géometries du fisc Byzantin*, P. Lethielleux, Paris 1991, p. 13. In Egypt, geometry was applied arithmetic. There were no systems for the production of rules, without that however, meaning that they ignored them. See Heath, *History of Greek Mathematics*, vol.1, p.124.

[6] The Byzantines had great respect for Euclidian geometry. Also N.Tartaglia in the West believed that it was a gift of God for anyone to be able to use the work of Euclid in mechanical applications and technologies. See P. L. Rose, *The Italian Renaissance of Mathematics*, librairie Droz, Genève 1975, p. 152,153.

[7] D. E. Smith, *History of Mathematics*, Dover, New York, 1958, vol.I, p.274.

[8] as above, p. 269.

[9] D. E. Smith, *History of Mathematics*, Dover, New York, 1958, vol.II, p.307.

[10] K. Vogel, "Leonardo Fibonacci", Dictionary of Scientific Biography, ed. Ch.Coulston, Gillispie, Ch. Scribner's sons, vol. IV, p.604–613.

marach@germanosnet.gr