Constructing the shortest path on a cylindrical surface

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Digital media warrant a reappraisal of established conceptual fields and a search for new ones densely providing access to powerful mathematical ideas. This study reports secondary students' meaning making around the notion of intrinsically defined curvature in space by means of a tool integrating programming, dynamic manipulation of variable values and a simulation of 3D space. The study involved 15 ninth grade school students' attempt to design the shortest path between two points on a cylindrical surface are presented in this paper. Camera perusal and zoom allow for a change of viewpoints of the constructed figure. The findings yield meanings around concepts notoriously difficult even in undergraduate mathematics, such as differential stereometry, limits and curvature as systematic trihedron state change.

Keywords: C*urvature, helix, stereometry, meaning-making, programmable media*

Introduction

Although curves appear in abundance in primary and secondary curricula, they are given the status of an auxiliary mathematical object to diverse structures either from geometry, e.g. circles-arcs, stereometry, e.g. cylinders-conic sections, or from algebra where the focus is of course on functions. Curvature is hardly discussed as a central notion, particularly in 3D space. Yet, in real physical space curves are truly abundant, in navigation they are key. Representations and notations from the pre-digital era are certainly one of the obstacles for students to access conceptual fields with curvature at their centre (we are intentionally using Vergnaud's construct, 1988). Here we use a digital medium integrating programmability with dynamic manipulation in simulated 3D space to get a sense of the meanings high school students may generate around differential curvature in space.

Curvature can be uniquely defined (apart from its position in space) by three elements of its arc, length, curvature and torsion (Lipschutz, 1969). The notion of curvature, the study of the properties of a curve and of the ways it can be approached consist one of the most important issues in tertiary education, as, for example, in differential geometry. The pre-digital formalism as well as the complicated formulas required consist a significant obstacle so that these notions and differential geometry in general can become approachable to many a student even at the tertiary level (Henderson, 1995; Kawski, 2003).

The encoding of the knowledge about curves, has historically gone through different stages. Euclid defines the curve as 'length without width' or 'end of a surface', without giving its definition in a general form restricted by general findings. But with the emergence of analytic geometry by Descartes, curves were defined as a mathematical sequence of points uniquely identified by two values. Later, the prevalence of the concept of function as a central concept in the curricula of secondary education, established functions as an umbrella under which large parts of mathematics can be interpreted. As a consequence the only curves introduced in secondary education are graphical representations, namely curves which are represented only as secondary data representations or equations. The appearance of Turtle Geometry (Papert, 1980) constituted a first but most significant suggestion to consider restructuring knowledge (Wilensky, 2010) about curvature.

Papert proposed an intrinsic approach to geometry as a way to use digital media to provide kids with access to powerful ideas in environments rich in opportunity for meaning making (Kynigos, 1993). The intrinsic definition of curve on the plane was thus by means of the 'turtle', the cybernetic programmable unit vector (heading, position, zero length), making alternative state changes with a value approaching zero. So, this geometry addresses the problem of the local description of a curve using the kinematic picture of the curve as the line resulting from position changes (Loethe, 1992). But what about curvature in space? The first digital tool to simulate programmable turtle geometry in space appeared relatively early by Reggini in 1985, so it may be surprising that there was no further epistemological or pedagogical analysis regarding curvature represented with this medium. In space, the intrinsic description of a curve can be achieved by using a mobile system of perpendicular vectors describing the tangent vector and the osculating plane of a curve. The turtle moves only in the direction of the nose and 'sits' on the osculating plane (Loethe, 1992, p.72). Rotation of the trihedron as it moves is given by the curvature and torsion. Precisely, as the rate of change of the tangent is characterized by the curvature, thus the rate of change of the osculating plane is characterized by the torsion of the curve (Aleksandrov et al., 1969, p.75). Our research group has been interested in identifying meanings generated by students around the field of intrinsically defined curvature on the plane, using a tool we developed (we called it 'Turtleworlds') to integrate programmable turtle geometry with dynamic manipulation of variable procedure values (Kynigos & Psycharis, 2003). In this paper we address meanings of intrinsic curvature in space with a new version of the tool which we now call 'MaLT-Turtlesphere' (Kynigos & Latsi, 2007) and start from giving students the problem of the shortest path between two points on a cylinder.

The theoretical frame

Vergnaud (1988), introduced the notion of conceptual field as a set of situations the mastering of which requires mastery of several concepts of different nature. He claims that “a single concept does not refer to only one type of situation, and a single situation cannot be analyzed with only one concept” (p. 141), and he argues that teachers and researchers should study conceptual fields rather than isolated concepts. In our study we wanted to study meaning making on curvature by giving students the problem of finding the shortest path between two points on a cylinder. We thus perceived the problem as belonging to the conceptual field of ‘curvature in space’ as the notions, for example, of rate of change and arc length which are involved in the procedure of designing a curve based on the polygonal approximation, are directly related to the notions of curvature and torsion in space. With our basic aim being to examine the meanings the students develop (Noss and Hoyles, 1996) in relation with the notions of differential geometry we designed activities based on constructionism (Kafai and Resnick, 1996). Students would engage in meaning making through bricolage with digital artefacts. In recent years, we have developed a pedagogical design construct and method where we start students off by providing them with a 'half-baked microworld' (Kynigos, 2007). It is a specially designed digital artefact with one or more built in bugs resulting in some faulty appearance and/or behavior when it is manipulated dynamically. It is designed to challenge students to decompose, change and debug the artefact and then construct something by using the correct version as a building block. Half-baked microworlds serve as starting points for the user to be acquainted with the ideas hidden behind the procedure of their construction.

The computational environment

The computational environment we used in our present research is MaLT-Turtlesphere (http://etl.uoa.gr/malt2) integrating Logo-based turtle geometry with dynamic manipulation of variable values resulting in DGS-like continuous change of the turtle figures at hand. This version of turtlesphere also afforded the insertion of stereometrical objects one of which was a cylinder, dynamically manipulable with respect to some key properties. The turtle movements are determined by following commands: fd(:n) and bk(:n) which command the turtle to take steps forwards or backwards, lt(:n) and rt(:n) move the turtle n degrees to the left or the right in its plane (osculating plane), and borrowed from Reggini's definition, dp(:n) and up(:n) turn the turtle upwards or downwards and rr(:n) , rl(:n) move the turtle around its axis. The basic tools of MaLT-Turtlesphere are the uni-dimensional variation tool (1DVT) which enables the user to dynamically manipulate the values of variables in a represented object and the 2d variation tool which is a two dimensional orthonormal system and is used to determine the co-variation of the values of two variables. An additional characteristic is its 3d Camera Controller which gives students the ability to dynamically manipulate the camera by means of the active vector and observes the object in the simulated 3d space from any side and direction he/she wishes. We should also point out the ability the user has got to insert ready-made 3d objects, such as a sphere or a cylinder, in a 3d virtual space and dynamically manipulate them.

The Problem

The students were given the following problem: ‘Calculate and design the shortest path between two points on a cylindrical surface’. Our students were informed that this program would enable them to work out the way they could design such a path and that, at the end, they themselves could use it in order to construct their own models. The students were told that they were allowed to use any method and materials they liked (for example, paper and scissors) and the following half-baked microworld under the name the ‘shortest path’:

to shortestpath :n :s :dx :c

repeat :n [rl(:s) lt(:c) fd(:dx)] end ('end' is on a separate line, placed here to save space).

This microworld comprises a program with four variables each of which express the following: n expresses a number of repetitions, s expresses the turning of the turtle around the directions of its path, dx defining the length of the turtle step and c defining the turning of the turtle in its plane. The execution of the above code produces a polygonal line (either in space or in plane) or a straight line. But in the case when dx is very small, three kinds of curves can result from the aforementioned microworld, with the characteristic of stability of proportions ‘turning and twisting relative to traveled space’. If s=0, then we have a curve on a plane. For s=0 and c=0 line segments. For s=0 and $c \neq 0$ circle arcs. Solving the problem requires finding the shortest path between two points on a cylindrical surface, which means that the target is achieved when dx tends to zero. This leads to limit procedures: $\lim_{dx \to 0} \frac{c}{dx} = k$, which gives the curvature of the circle. If s is different from zero, helical lines are generated in space and similar conclusions are drawn (in case c = 0, a straight line arises). So, this code for creating a helix around the surface of a specific cylinder is half-baked in that it does not contain the property of each of the two turns being a function of the value of displacement (fd) and that the value of dx needs to tend to approach zero. If in the preceding code, we replace arguments with suitable functions and introduce a tail recursion, any line in space can occur. For example, if we replace the arguments of turning and twisting with trigonometric functions, a closed curve in space can occur.

The Method

We adopted a design research method (Cobb et al., 2003). In this paper we discuss part of a broader research, which was developed in three phases: the first phase involving two students 3rd grade secondary school, in the second phase with the participation 15 students (a class 3rd grade secondary school) and which lasted 24 hours, and finally, in the third phase involving five higher education's students. These particular students of the second phase had already been familiarized with constructions in the logo programming language in the Turtleworlds environment. A sound and picture software (HyperCam 2) was used to record data and enabled the researcher to record the students’ actions and the conversations amongst the participants. In order to analyze the students’ mathematical thinking we were interested in the ways the students interacted with the available components of the software and in the ways they constructed mathematical meanings. We centrally used the notions of meaning making and situated abstractions, which enabled us to describe how the students construct mathematical meanings based on the functions of the particular software they were using and on the conversations between them (Noss & Hoyles, 1996). We also found the construct of 'instrumentalization' taken from the theory of instrumental genesis (Guin and Trouche, 1999, Kynigos & Psycharis, 2013) helpful in showing us was how the students were trying to change the functionalities of the ‘faulty’ microworld they were given aiming to produce a different artefact which automatically gives a circle and a helix with the shortest length.

Findings

The circle approach through limiting curvature

Even if the majority of students at first turned to the software they had been given in their effort to give an answer, they soon realized something else should be done first to make sense of the problem. They decided to use tangible objects first, paper, pen and the scissors, they had also been given. By selecting two points on the cylindrical surface and then rolling a piece of paper to form a cylinder and un-rolling it, they came to the conclusion that the shortest path could be a circle, a helical or a straight line. Upon un-rolling the cylinder they noticed that the line which was formed would be a straight line on the plane (geodesic in plane) but when they re–rolled up the cylinder, a helix or a circle was formed. Nevertheless, this conclusion, although it seemed to be the solution, did not seem to satisfy the students at all. Here is a typical answer from two students:

Student 1: If we could suppose that the cylinder opens, then okay it is a straight line
Student 2: But if the cylinder could not open? (Meaning: then how could we design the helix?)

They then started exploring the half-baked code firstly by dragging the variable values. All students decided to focus first on getting the code to create a circle around a fixed cylinder. Some kept the value of the rl command to zero, some decided to chuck it out of the procedure, starting to work on the formalism. The students at hand took the latter option and tried out dragging to understand the behavior of the turtle path (Brunström & Fahlgren, 2015). For this circle, a common technique was the winding of the polygonal line at a constant circle or at the bottom of an inserted cylinder from the software library. The completion of the first winding lead students to put values dx =1, c = 29. But when the researcher asked the question about the kind of path that was formed, students concluded by zooming that it was a polygonal line, and a further reduction of dx was needed. Students, with the help of changes decreased the value of dx from dx = 1 to dx = 0.1, and then did the same for dx = 0.01, while modifying the value of c as well, as the polygonal line continued to wind in a solid circle. Their attempts brought them to conclude that the turn value should be dependent on the displacement value if the turtle trace was to be a good fit to the base of the given cylinder. They then decided that the code should contain a proportional relation of the variables c and dx, and modified the half-baked microworld engaging in an intrumentalization activity. The result for these students and, as it turned out, for the majority of the participants, was a code like the following, with a differentiation in the arguments of the turtle turn:

```
to shortestpath  :n  :dx
repeat :n  [lt(29*:dx)  fd(:dx)] end
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The dialogue continued yielding that the students considered the circle as a polygon with sides that are constantly decreasing in length:

Researcher: so, for which rates do you get the requested circle?
Student 1: for small dx, for example 0.1
Researcher: ie for dx = 0.1 will we have a circle?
Student 2: we will have a polygon
Researcher: and which may be the required rates?
Student 2: we can’t be exact because as we put smaller numbers, it will be approaching the solid circle (at the same time the student zoom and manipulate the slider of dx to continuously lower numbers to prove their claim)

The students' efforts show a change in the way they thought about curvature, starting from a static approach (with dx equaling a constant value of 1) to a dynamic (the more dx diminishes, the better approach to curvature). This was evident in their correction of the code initially achieved with dx to be small (usually teams chose for dx a tenth or hundredth approach). Thus, initially the circle formed by the mean curvature (c / dx = constant) determined the forward movement of the turtle in relation to the dx. This instrumentalization action resulted in a modification of the shortest path code and provided us researchers with a lens to students' development of a situated abstraction on the concept of curvature. The problem that was given to find the shortest path, thus led them at first to think of curvature as a limit and the circle as resulting from a limiting process and not simply by dx small. Although a strictly symbolic form of a limit was unknown to the students, the role of the limit process seemed to be played by the slider of dx.

From a static to a dynamic aspect of the helix

For the construction of the helix with Turtlesphere, students at first could not implement a technical approach as in the circle, since there was not a preplanned helix on the cylindrical surface. So they resorted to properties discovered during the deformation process of the flat surface and the situated abstraction for the notion of helix which was delivered by them as follows: *'helix is a curve that is wrapped in a cylindrical surface and if it unfolds, a straight line emerges'.* The designing of such a curve though without the use of tangible materials, and the ability to generalize such a procedure demand the use of differential geometry notions which reflect the Frenet-Serret frame movement in space. The students appeared to realize the limitations of tangible materials, and the inability to generalize the procedure in situations when their use is impossible.

The students' speculation stimulated the researcher to turn their attention to the half–baked code they had already had at their disposal. The students chose again to insert a model cylinder with specific dimensions, and by dragging the variation tools they tried to achieve the construction of a helical line which twisted round the cylinder with its two ends being the ends of the generator of the cylinder with the above characteristics. Their initial suppositions referred to values which, although at first sight seemed to have achieved their goal (that is the helical line to twist round the cylinder), the use of the perusal camera proved wrong. Thus, from that time on each and every attempt of theirs initially comprised finding the values for n, c, s and dx with the simultaneous use of the camera and change of the values of the variables. A group of students, at their first correct attempt (with dx=1), came to the following values: n=14, c=25, s=5 and dx=1. Although they seemed to be satisfied with the result of their experimentations, they continued to experiment after the following questions on the researcher's part:

Researcher: Is this a helix? (They play with the camera, zooming in at the same time)
Student 1: They look like lots of straight lines (they are referring to the line segments which the helical line is composed of and with the execution of the half –baked microworld provides them with)
Researcher: What can you do so that you can turn it into a helix?
Student 1: Eliminate the angles
Researcher: How can you eliminate the angles?
Student 1: If we decrease dx, let's say to 0.1

Students' instrumentalization initially started by dragging the slider of dx and with the help of graphical feedback. Since dragging dx to dx = 1 gave a polygonal line and not the curve as the paper-folding approach, the students started to drag the slider of s. Their attention now concerned the discovery of the relationship between the three variables (c, dx, s) and the consequent changes to the half-baked code. The situated abstractions that this particular group of students seemed to have built arise from the need to prevent the distortion of the figure and satisfy the 'definition' that had created for the concept of helix based on the paper- folding approach. As dx took smaller and smaller values a line was given which looks like a helix with a length constantly decreasing and that the ratios c/dx and s/dx remain invariant and equal to 25 and 5 respectively. In fact, the rate of change of directions of the segments the turtle is moving on and its plane remain invariant. The replacement of the ratios they discovered in their initial code provided them with what they claimed was the 'correct' code and the solution in demand:

```
to shortestpath :n :dx
repeat :n [rl(5*:dx)  lt(25*:dx)  fd(:dx)]  end
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Researcher: Which values provide us with the helix we are looking for?

Student 1: The smaller dx is the better.

The students seemed to realize that the solution they were looking for did not only consist of the specific values of the variables but it should also combine a limited procedure for dx.

Conclusions

The purpose of the present research was dual: Firstly, to study the degree to which this particular digital tool and microworld could form the basis for secondary level students to study notions in the conceptual field of curvature in space and secondly, to study the meanings developed by these particular students in their attempt to design the shortest path between two points on a cylindrical surface. The students expressed mathematical meanings for a number of notions of differential calculus (rate of change, limit) as well as of differential geometry (curvature, torsion and geodesic) which has been shown to be notions difficult to be approached by even math students. One of the major advantages of the method applied is the fact that, not only were students able to visualize the way a normal curve is constructed by the motion of a movable trihedron in space (the role of which was replaced by the 'turtle') but the students were also given the ability to study, explore and symbolically represent these movements via programming and dynamic manipulation. For example, the circle is constructed by the turtle avatar with the characteristic of working stability, and not just through the stability of the ratio c / dx, i.e. the curvature formula in Logo. The dynamic manipulation resulting in figural change helped the students focus on the limiting process of this ratio which reflects the notion of curvature. The students changed their conception of helix from a static approach to a dynamic aspect, i.e. as a line made from an avatar with the characteristic of stability in both its turning and rotating around the line of motion. Although the way they used to design the helix did not tally with the strict formalism of differential geometry, the answers the the meanings they generated are indicative of the fact that a restructuration of the notion of curve relying on concepts of curvature and torsion, and with the turtle replacing the role of the moving trihedron to create a curve in space, is feasible in secondary education.

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References

Aleksandrov, A., Kolmogorov, A., & Lavrent’ev, M. (1969), Mathematics: Its Content, Methods, and Meaning, Vol. II (2nd ed.), Cambridge: MIT Press, Brunström, M., Fahlgren., M (2015) Using slider tools to explore and validate. Konrad Krainer; Naďa Vondrová. *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education*, Prague, Czech Republic. pp.2588-2589, Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education. <hal-01289391>

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, Vol. 32-1, 9-13.

Guin, D. & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: the case of calculators. I.J.C.M.L., 3(3): 195–227.

Henderson, D (1996). "Alive Mathematical Reasoning", Proceedings, Canadian Mathematics Education Study Group, Halifax, NS: Mount Saint Vincent University Press, 27-33,

Kafai, Y. & Resnick, M. (eds.) (1996). Constructionism in practice: Designing, thinking and learning in a digital world. Lawrence Erlbaum Publishers, Mahwah

Kawski, M (2003). Curvature for everyone. 8th Asian Technology Conf Math, Taiwan 2003,

Kynigos, C. & Latsi, M. (2007). Turtle's navigation and manipulation of geometrical figures constructed by variable processes in 3d simulated space. Informatics in Education, Vol. 6, No. 2, 359-372.

Kynigos, C. & Psycharis, G. (2003). 13 Year olds' Meanings Around Intrinsic Curves with a Medium for Symbolic Expression and Dynamic Manipulation. *Proceedings of the 27th Psychology of Mathematics Education Conference* (3, 165-172), Paterman, N., Dougherty, B., Zilliox, J. (eds), CRDG, College of Education, University of Hawai'I, Honolulu, U.S.A.

Kynigos, C. (1993). Children's Inductive Thinking During Intrinsic and Euclidean Geometrical Activities in a Computer Programming Environment, *Educational Studies in Mathematics*, vol. 24, 177-197.

Kynigos, C. (2007). Half-Baked Logo Microworlds as Boundary Objects in Integrated Design, Informatics in Education, 2007, Vol. 6, (2), 1–24.

Lipschutz, M. M. (1969). Theory and Problems of Differential Geometry (Schaum's Outline Series). New York: McGraw-Hill

Loethe H., (1992). Conceptually Defined Turtles, in: (Hoyles, D. and Noss, R. eds.),Learning Mathematics and Logo, The MIT Press,

Noss,R., & Hoyles, C. (1996). Windows on mathematical meanings. Dordrecht, Kluwer

Papert, S. (1980). Mindstorms: children, computers and powerful ideas. New York: Basic Books

Reggini, H. C. (1985). *Ideas y formas: Explorando el espacio con Logo*. Buenos Aires: Galápago

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 141-161). Hillsdale, NJ: Lawrence Erlbaum Associates.

Wilensky, U. (2010). Restructurations: Reformulating knowledge disciplines throw new representational forms. In J. E. Clayson & I.Kalas (Ed), constructionism 2010 Paris, France.