61. Approximate $e^{0.2}$ using the first five terms of the series. Compare this approximation with your calculator evaluation of $e^{0.2}$.

62. Approximate $e^{-0.5}$ using the first five terms of the series. Compare this approximation with your calculator evaluation of $e^{-0.5}$.

63. Show that: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

64. Show that: $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

SECTION 8-2 Mathematical Induction

- Introduction
- Mathematical Induction
- Additional Examples of Mathematical Induction
- Three Famous Problems

• Introduction

In common usage, the word "induction" means the generalization from particular cases or facts. The ability to formulate general hypotheses from a limited number of facts is a distinguishing characteristic of a creative mathematician. The creative process does not stop here, however. These hypotheses must then be proved or disproved. In mathematics, a special method of proof called **mathematical induction** ranks among the most important basic tools in a mathematician's toolbox. In this section mathematical induction will be used to prove a variety of mathematical statements, some new and some that up to now we have just assumed to be true.

We illustrate the process of formulating hypotheses by an example. Suppose we are interested in the sum of the first n consecutive odd integers, where n is a positive integer. We begin by writing the sums for the first few values of n to see if we can observe a pattern:

$$1 = 1 \qquad n = 1$$

$$1 + 3 = 4 \qquad n = 2$$

$$1 + 3 + 5 = 9 \qquad n = 3$$

$$1 + 3 + 5 + 7 = 16 \qquad n = 4$$

$$1 + 3 + 5 + 7 + 9 = 25 \qquad n = 5$$

Is there any pattern to the sums 1, 4, 9, 16, and 25? You no doubt observed that each is a perfect square and, in fact, each is the square of the number of terms in the sum. Thus, the following conjecture seems reasonable:

Conjecture P_n: For each positive integer n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

That is, the sum of the first n odd integers is n^2 for each positive integer n.

So far ordinary induction has been used to generalize the pattern observed in the first few cases listed above. But at this point conjecture P_n is simply that—a conjecture. How do we prove that P_n is a true statement? Continuing to list specific cases will never provide a general proof—not in your lifetime or all your descendants' lifetimes! Mathematical induction is the tool we will use to establish the validity of conjecture P_n.

Before discussing this method of proof, let's consider another conjecture:

Conjecture Q_n: For each positive integer n, the number $n^2 - n + 41$ is a prime number.

TABLE 1

n	$n^2 - n + 41$	Prime?
1	41	Yes
2	43	Yes
3	47	Yes
4	53	Yes
5	61	Yes

It is important to recognize that a conjecture can be proved false if it fails for only one case. A single case or example for which a conjecture fails is called a **counterexample.** We check the conjecture for a few particular cases in Table 1. From the table, it certainly appears that conjecture Q_n has a good chance of being true. You may want to check a few more cases. If you persist, you will find that conjecture Q_n is true for n up to 41. What happens at $n = 41$?

$$41^2 - 41 + 41 = 41^2$$

which is not prime. Thus, since $n = 41$ provides a counterexample, conjecture Q_n is false. Here we see the danger of generalizing without proof from a few special cases. This example was discovered by Euler (1707–1783).

EXPLORE-DISCUSS 1 Prove that the following statement is false by finding a counterexample: If $n \geq 2$, then at least one-third of the positive integers less than or equal to n are prime.

• Mathematical Induction

We begin by stating the *principle of mathematical induction*, which forms the basis for all our work in this section.

Theorem 1

Principle of Mathematical Induction

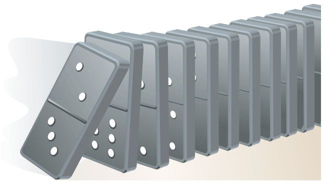
Let P_n be a statement associated with each positive integer n, and suppose the following conditions are satisfied:

1. P_1 is true.
2. For any positive integer k, if P_k is true, then P_{k+1} is also true.

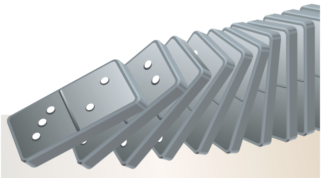
Then the statement P_n is true for all positive integers n.

Theorem 1 must be read very carefully. At first glance, it seems to say that if we assume a statement is true, then it is true. But that is not the case at all. If the two conditions in Theorem 1 are satisfied, then we can reason as follows:

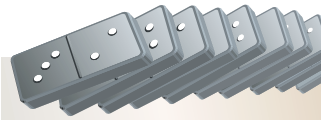
P_1 is true.	Condition 1
P_2 is true, because P_1 is true.	Condition 2
P_3 is true, because P_2 is true.	Condition 2
P_4 is true, because P_3 is true.	Condition 2
⋮	⋮



Condition 1: The first domino can be pushed over.
(a)



Condition 2: If the kth domino falls, then so does the $(k + 1)$st.
(b)



Conclusion: All the dominoes will fall.
(c)

FIGURE 1 Interpreting mathematical induction.

Since this chain of implications never ends, we will eventually reach P_n for any positive integer n.

To help visualize this process, picture a row of dominoes that goes on forever (see Fig. 1) and interpret the conditions in Theorem 1 as follows: Condition 1 says that the first domino can be pushed over. Condition 2 says that if the kth domino falls, then so does the $(k + 1)$st domino. Together, these two conditions imply that all the dominoes must fall.

Now, to illustrate the process of proof by mathematical induction, we return to the conjecture P_n discussed earlier, which we restate below:

$$P_n\colon \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad n \text{ any positive integer}$$

We already know that P_1 is a true statement. In fact, we demonstrated that P_1 through P_5 are all true by direct calculation. Thus, condition 1 in Theorem 1 is satisfied. To show that condition 2 is satisfied, we assume that P_k is a true statement:

$$P_k\colon 1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Now we must show that this assumption implies that P_{k+1} is also a true statement:

$$P_{k+1}\colon \quad 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Since we have assumed that P_k is true, we can perform operations on this equation. Note that the left side of P_{k+1} is the left side of P_k plus $(2k + 1)$. So we start by adding $(2k + 1)$ to both sides of P_k:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \qquad P_k$$

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

Add $2k + 1$ to both sides.

Factoring the right side of this equation, we have

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \qquad P_{k+1}$$

But this last equation is P_{k+1}. Thus, we have started with P_k, the statement we assumed true, and performed valid operations to produce P_{k+1}, the statement we want to be true. In other words, we have shown that if P_k is true, then P_{k+1} is also true. Since both conditions in Theorem 1 are satisfied, P_n is true for all positive integers n.

• Additional Examples of Mathematical Induction

Now we will consider some additional examples of proof by induction. The first is another summation formula. Mathematical induction is the primary tool for proving that formulas of this type are true.

EXAMPLE 1 Proving a Summation Formula

Prove that for all positive integers n

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Proof State the conjecture:

$$P_n: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Part 1 Show that P_1 is true.

$$P_1: \quad \frac{1}{2} = \frac{2^1 - 1}{2^1}$$
$$= \frac{1}{2}$$

Thus, P_1 is true.

Part 2 Show that if P_k is true, then P_{k+1} is true. It is a good practice to always write out both P_k and P_{k+1} at the beginning of any induction proof to see what is assumed and what must be proved:

$$P_k: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

We assume P_k is true.

$$P_{k+1}: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

We must show that P_{k+1} follows from P_k.

We start with the true statement P_k, add $1/2^{k+1}$ to both sides, and simplify the right side:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k} \qquad P_k$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \mathbf{\frac{1}{2^{k+1}}} = \frac{2^k - 1}{2^k} + \mathbf{\frac{1}{2^{k+1}}}$$
$$= \frac{2^k - 1}{2^k} \cdot \frac{2}{2} + \frac{1}{2^{k+1}}$$
$$= \frac{2^{k+1} - 2 + 1}{2^{k+1}}$$
$$= \frac{2^{k+1} - 1}{2^{k+1}}$$

Thus,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}} \qquad P_{k+1}$$

and we have shown that if P_k is true, then P_{k+1} is true.

Conclusion Both conditions in Theorem 1 are satisfied. Thus, P_n is true for all positive integers n.

Matched Problem 1 Prove that for all positive integers n

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

The next example provides a proof of a law of exponents that previously we had to assume was true. First we redefine a^n for n a positive integer, using a recursion formula:

DEFINITION 1

Recursive Definition of a^n

For n a positive integer

$$a^1 = a$$
$$a^{n+1} = a^n a \qquad n \geq 1$$

EXAMPLE 2 **Proving a Law of Exponents**

Prove that $(xy)^n = x^n y^n$ for all positive integers n.

Proof State the conjecture:

$$P_n: \quad (xy)^n = x^n y^n$$

Part 1 Show that P_1 is true.

$$\begin{aligned} (xy)^1 &= xy && \text{Definition 1} \\ &= x^1 y^1 && \text{Definition 1} \end{aligned}$$

Thus, P_1 is true.

Part 2 Show that if P_k is true, then P_{k+1} is true.

$$\begin{aligned} P_k: &\quad (xy)^k = x^k y^k && \text{Assume } P_k \text{ is true.} \\ P_{k+1}: &\quad (xy)^{k+1} = x^{k+1} y^{k+1} && \text{Show that } P_{k+1} \text{ follows from } P_k. \end{aligned}$$

Here we start with the left side of P_{k+1} and use P_k to find the right side of P_{k+1}:

$$\begin{aligned} (xy)^{k+1} &= (xy)^k (xy)^1 && \text{Definition 1} \\ &= x^k y^k xy && \text{Use } P_k\text{: } (xy)^k = x^k y^k \\ &= (x^k x)(y^k y) && \text{Property of real numbers} \\ &= x^{k+1} y^{k+1} && \text{Definition 1} \end{aligned}$$

Thus, $(xy)^{k+1} = x^{k+1}y^{k+1}$, and we have shown that if P_k is true, then P_{k+1} is true.

Conclusion Both conditions in Theorem 1 are satisfied. Thus, P_n is true for all positive integers n.

Matched Problem 2 Prove that $(x/y)^n = x^n/y^n$ for all positive integers n.

Our last example deals with factors of integers. Before we start, recall that an integer p is *divisible* by an integer q if $p = qr$ for some integer r.

EXAMPLE 3 Proving a Divisibility Property

Prove that $4^{2n} - 1$ is divisible by 5 for all positive integers n.

Proof Use the definition of divisibility to state the conjecture as follows:

$$P_n\text{:}\quad 4^{2n} - 1 = 5r \qquad \text{for some integer } r$$

Part 1 Show that P_1 is true.

$$P_1\text{:}\quad 4^2 - 1 = 15 = 5 \cdot 3$$

Thus, P_1 is true.

Part 2 Show that if P_k is true, then P_{k+1} is true.

$$P_k\text{:}\quad 4^{2k} - 1 = 5r \qquad \text{for some integer } r \qquad \text{Assume } P_k \text{ is true.}$$
$$P_{k+1}\text{:}\quad 4^{2(k+1)} - 1 = 5s \qquad \text{for some integer } s \qquad \text{Show that } P_{k+1} \text{ must follow.}$$

As before, we start with the true statement P_k:

$$4^{2k} - 1 = 5r \qquad P_k$$
$$4^2(4^{2k} - 1) = 4^2(5r) \qquad \text{Multiply both sides by } 4^2.$$
$$4^{2k+2} - 16 = 80r \qquad \text{Simplify.}$$
$$4^{2(k+1)} - 1 = 80r + 15 \qquad \text{Add 15 to both sides.}$$
$$= 5(16r + 3) \qquad \text{Factor out 5.}$$

Thus,

$$4^{2(k+1)} - 1 = 5s \qquad P_{k+1}$$

where $s = 16r + 3$ is an integer, and we have shown that if P_k is true, then P_{k+1} is true.

Conclusion Both conditions in Theorem 1 are satisfied. Thus, P_n is true for all positive integers n.

Matched Problem 3 Prove that $8^n - 1$ is divisible by 7 for all positive integers n.

In some cases, a conjecture may be true only for $n \geq m$, where m is a positive integer, rather than for all $n \geq 0$. For example, see Problems 49 and 50 in Exercise 8-2. The principle of mathematical induction can be extended to cover cases like this as follows:

Theorem 2

Extended Principle of Mathematical Induction

Let m be a positive integer, let P_n be a statement associated with each integer $n \geq m$, and suppose the following conditions are satisfied:

1. P_m is true.

2. For any integer $k \geq m$, if P_k is true, then P_{k+1} is also true.

Then the statement P_n is true for all integers $n \geq m$.

• Three Famous Problems

The problem of determining whether a certain statement about the positive integers is true may be extremely difficult. Proofs may require remarkable insight and ingenuity and the development of techniques far more advanced than mathematical induction. Consider, for example, the famous problems of proving the following statements:

1. **Lagrange's Four Square Theorem, 1772:** Each positive integer can be expressed as the sum of four or fewer squares of positive integers.
2. **Fermat's Last Theorem, 1637:** For $n > 2$, $x^n + y^n = z^n$ does not have solutions in the natural numbers.
3. **Goldbach's Conjecture, 1742:** Every positive even integer greater than 2 is the sum of two prime numbers.

The first statement was considered by the early Greeks and finally proved in 1772 by Lagrange. Fermat's last theorem, defying the best mathematical minds for over 350 years, finally succumbed to a 200-page proof by Prof. Andrew Wiles of Princeton University in 1993. To this date no one has been able to prove or disprove Goldbach's conjecture.

EXPLORE-DISCUSS 2

(A) Explain the difference between a theorem and a conjecture.

(B) Why is "Fermat's last theorem" a misnomer? Suggest more accurate names for the result.

Answers to Matched Problems

1. Sketch of proof. State the conjecture: P_n: $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$

Part 1. $1 = \frac{1(1 + 1)}{2}$. P_1 is true.

Part 2. Show that if P_k is true, then P_{k+1} is true.

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2} \qquad P_k$$

$$1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

$$= \frac{(k + 1)(k + 2)}{2} \qquad P_{k+1}$$

Conclusion: P_n is true.

2. Sketch of proof. State the conjecture: P_n: $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

Part 1. $\left(\frac{x}{y}\right)^1 = \frac{x}{y} = \frac{x^1}{y^1}$. P_1 is true.

Part 2. Show that if P_k is true, then P_{k+1} is true.

$$\left(\frac{x}{y}\right)^{k+1} = \left(\frac{x}{y}\right)^k\left(\frac{x}{y}\right) = \frac{x^k}{y^k}\left(\frac{x}{y}\right) = \frac{x^k x}{y^k y} = \frac{x^{k+1}}{y^{k+1}}$$

Conclusion: P_n is true.

3. Sketch of proof. State the conjecture: P_n: $8^n - 1 = 7r$ for some integer r

Part 1. $8^1 - 1 = 7 = 7 \cdot 1$. P_1 is true.

Part 2. Show that if P_k is true, then P_{k+1} is true.

$$8^k - 1 = 7r \qquad P_k$$

$$8(8^k - 1) = 8(7r)$$

$$8^{k+1} - 1 = 56r + 7 = 7(8r + 1) = 7s \qquad P_{k+1}$$

Conclusion: P_n is true.

EXERCISE 8-2

A

In Problems 1–4, find the first positive integer n that causes the statement to fail.

1. $3^n + 4^n \geq 5^n$

2. $n^2 - 3n < 100$

3. $17^n - 1$ is divisible by 2^n

4. $n^2 = 5n - 6$

Verify each statement P_n in Problems 5–10 for $n = 1, 2,$ and 3.

5. P_n: $2 + 6 + 10 + \ldots + (4n - 2) = 2n^2$

6. P_n: $4 + 8 + 12 + \ldots + 4n = 2n(n + 1)$

7. P_n: $a^5 a^n = a^{5+n}$

8. P_n: $(a^5)^n = a^{5n}$

9. P_n: $9^n - 1$ is divisible by 4

10. P_n: $4^n - 1$ is divisible by 3

Write P_k and P_{k+1} for P_n as indicated in Problems 11–16.

11. P_n in Problem 5

12. P_n in Problem 6

13. P_n in Problem 7

14. P_n in Problem 8

15. P_n in Problem 9

16. P_n in Problem 10

In Problems 17–22, use mathematical induction to prove that each P_n holds for all positive integers n.

17. P_n in Problem 5

18. P_n in Problem 6

19. P_n in Problem 7

20. P_n in Problem 8

21. P_n in Problem 9

22. P_n in Problem 10

B

In Problems 23–26, prove the statement is false by finding a counterexample.

23. If $n > 2$, then any polynomial of degree n has at least one real zero.

24. Any positive integer $n > 7$ can be written as the sum of three or fewer squares of positive integers.

25. If n is a positive integer, then there is at least one prime number p such that $n < p < n + 6$.

26. If a, b, c, d are positive integers such that $a^2 + b^2 = c^2 + d^2$, then $a = c$ or $a = d$.

In Problems 27–42, use mathematical induction to prove each proposition for all positive integers n, unless restricted otherwise.

27. $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$

28. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

29. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$

30. $1 + 8 + 16 + \cdots + 8(n - 1) = (2n - 1)^2;\ n > 1$

31. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

32. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

33. $\frac{a^n}{a^3} = a^{n-3};\ n > 3$

34. $\frac{a^5}{a^n} = \frac{1}{a^{n-5}};\ n > 5$

35. $a^m a^n = a^{m+n};\ m, n \in N$
[*Hint:* Choose m as an arbitrary element of N, and then use induction on n.]

36. $(a^n)^m = a^{mn};\ m, n \in N$

37. $x^n - 1$ is divisible by $x - 1$; $x \neq 1$
[*Hint:* Divisible means that $x^n - 1 = (x - 1)Q(x)$ for some polynomial $Q(x)$.]

38. $x^n - y^n$ is divisible by $x - y$; $x \neq y$

39. $x^{2n} - 1$ is divisible by $x - 1$; $x \neq 1$

40. $x^{2n} - 1$ is divisible by $x + 1$; $x \neq -1$

41. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$
[*Hint:* See Matched Problem 1 following Example 1.]

42. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n + 1)(n + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)}$

C

In Problems 43–46, suggest a formula for each expression, and prove your hypothesis using mathematical induction, $n \in N$.

43. $2 + 4 + 6 + \cdots + 2n$

44. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)}$

45. The number of lines determined by n points in a plane, no three of which are collinear

46. The number of diagonals in a polygon with n sides

In Problems 47–50, prove the statement is true for all integers n as specified.

47. $a > 1 \Rightarrow a^n > 1;\ n \in N$

48. $0 < a < 1 \Rightarrow 0 < a^n < 1;\ n \in N$

49. $n^2 > 2n;\ n \geq 3$

50. $2^n > n^2;\ n \geq 5$

51. Prove or disprove the generalization of the following two facts:

$$3^2 + 4^2 = 5^2$$
$$3^3 + 4^3 + 5^3 = 6^3$$

52. Prove or disprove: $n^2 + 21n + 1$ is a prime number for all natural numbers n.

If $\{a_n\}$ and $\{b_n\}$ are two sequences, we write $\{a_n\} = \{b_n\}$ if and only if $a_n = b_n$, $n \in N$. In Problems 53–56, use mathematical induction to show that $\{a_n\} = \{b_n\}$.

53. $a_1 = 1, a_n = a_{n-1} + 2;\ b_n = 2n - 1$

54. $a_1 = 2, a_n = a_{n-1} + 2;\ b_n = 2n$

55. $a_1 = 2, a_n = 2^2 a_{n-1};\ b_n = 2^{2n-1}$

56. $a_1 = 2, a_n = 3a_{n-1};\ b_n = 2 \cdot 3^{n-1}$

SECTION 8-3 Arithmetic and Geometric Sequences

- Arithmetic and Geometric Sequences
- nth-Term Formulas
- Sum Formulas for Finite Arithmetic Series
- Sum Formulas for Finite Geometric Series
- Sum Formula for Infinite Geometric Series