<u>Different kinds of Mathematical Induction</u>

(1) Mathematical Induction

Given $A \subset \mathbf{N}$, $[1 \in A \wedge (a \in A \Rightarrow a+1 \in A)] \Rightarrow A = \mathbf{N}$.

(2) (First) Principle of Mathematical Induction

Let P(x) be a proposition (open sentence), if we put

$A = \{x : x \in N \wedge p(x) \text{ is true}\}$ in (1), we get the Principle of Mathematical Induction.

If (1) P(1) is true;

(2) P(k) is true for some $k \in \mathbf{N}$ $\Rightarrow$ P(k+1) is true

then P(n) is true $\forall\, n \in \mathbf{N}$.

(3) Second Principle of Mathematical Induction

If (1) P(1) is true;

(2) $\forall\, 1 \le i \le k$, P(i) is true [i.e. $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ is true]

$\Rightarrow$ P(k+1) is true

then P(n) is true $\forall\, n \in \mathbf{N}$.

(4) Second Principle of Mathematical Induction (variation)

If (1) $P(1) \wedge P(2)$ is true;

(2) $P(k-1) \wedge P(k)$ is true for some $k \in \mathbf{N}\backslash\{1\}$ $\Rightarrow$ P(k+1) is true

then P(n) is true $\forall\, n \in \mathbf{N}$.

(5) Second Principle of Mathematical Induction (variation)

If (1) $P(1) \wedge P(2) \wedge \ldots \wedge P(m)$ is true;

(2) P(k) is true for some $k \in \mathbf{N} \Rightarrow$ P(k+m) is true

then P(n) is true $\forall\, n \in \mathbf{N}$.

(6) Odd-even M.I.

If (1) $P(1) \wedge P(2)$ is true;

(2) P(k) is true for some $k \in \mathbf{N} \Rightarrow$ P(k+2) is true

then P(n) is true $\forall\, n \in \mathbf{N}$.

More difficult types of Mathematical Induction

(7) Backward M.I.

If (1) P(n) is true $\forall$ n $\in$ A, where A is an infinite subset of **N**;

(2) P(k) is true for some k $\in$ **N** $\Rightarrow$ P(k–1) is true

then P(n) is true $\forall$ n $\in$ **N**.

(8) Backward M.I. (variation) (more easily applied than (7))

If (1) P(1) is true;

(2) $P(2^k)$ is true for some k $\in$ **N** $\Rightarrow$ $P(2^{k+1})$ is true;

(3) P(k) is true for some k $\in$ **N** $\Rightarrow$ P(k–1) is true

then P(n) is true $\forall$ n $\in$ **N**.

(9) Different starting point

If (1) P(a) is true, where a $\in$ **N**;

(2) P(k) is true for some k $\in$ **N,** where k $\geq$ a $\Rightarrow$ P(k+1) is true

then P(n) is true $\forall$ n $\in$ **N\{1, 2, …., a – 1 }**.

(10) Spiral M.I.

If (1) P(1) is true;

(2) P(k) is true for some k $\in$ **N** $\Rightarrow$ Q(k) is true

Q(k) is true for some k $\in$ **N** $\Rightarrow$ P(k+1) is true

then P(n) , Q(n) are true $\forall$ n $\in$ **N**.

(11) Double M.I.

Double M.I. involves a proposition P(m, n) with two variables m, n.

If (1) P(m, 1) and P(1, n) is true $\forall$ m, n $\in$ **N**;

(2) P(m+1, n) and P(m, n+1) are true for some m, n$\in$ **N**

$\Rightarrow$ P(m+1, n+1) is true

then P(m, n) is true $\forall$ m, n $\in$ **N.**

A Prime Number Theorem **[Second Principle of Mathematical Induction]**

Prove that the nth prime number $p_n < 2^{2^n}$.

Solution

Let P(n) be the proposition : $p_n < 2^{2^n}$.

For P(1), $p_1 = 2 < 2^{2^1}$ $\therefore$ P(1) is true.

Assume P(i) is true $\forall$ i s.t. $1 \le i \le k$, i.e. $p_1 < 2^{2^1}$, $p_2 < 2^{2^2}$,..., $p_k < 2^{2^k}$ …. (*)

For P(k + 1), Multiply all inequalities in (*), $p_1 p_2 \dots p_k < 2^{2^1} 2^{2^2} \dots 2^{2^k}$

$$p_1 p_2 \dots p_k + 1 \le 2^{2^1} 2^{2^2} \dots 2^{2^k} = 2^{2^1 + 2^2 + \dots + 2^k} = 2^{2^{k+1} - 2} < 2^{2^{k+1}}$$

$\therefore$ For any prime factor p of $p_1 p_2 \dots p_k + 1$, we have $p < 2^{2^{k+1}}$.

Since $p_1, p_2, \dots, p_k$ are not prime factor of $p_1 p_2 \dots p_k + 1$, we have $p_k < p$ and hence $p_{k+1} \le p$.

$\therefore$ $p_{k+1} \le p < 2^{2^{k+1}}$ $\therefore$ P(k + 1) is true.

By the Second Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbb{N}$.

Recurrive formula **[Second Principle of Mathematical Induction]**

Let $\{a_n\}$ be a sequence of real numbers satisfying $a_1 = 2$, $a_2 = 3$ and $a_{n+2} = 3a_{n+1} - 2a_n$.

Prove that $a_n = 2^{n-1} + 1$.

Solution

Let P(n) be the proposition : $a_n = 2^{n-1} + 1$.

For $P(1) \wedge P(2)$, $a_1 = 2 = 2^{1-1} + 1$, $a_2 = 2 = 2^{2-1} + 1$. $\therefore$ $P(1) \wedge P(2)$ is true.

Assume $P(k) \wedge P(k+1)$ is true for some $k \in \mathbb{N}$.

i.e. $a_k = 2^{k-1} + 1$ …. (1)

$a_{k+1} = 2^k + 1$ …. (2)

For P(k+2), $a_{k+2} = 3a_{k+1} - 2a_k = 3(2^k + 1) - 2(2^{k-1} + 1) = 2^{k+1} + 1$

$\therefore$ P(k + 2) is true.

By the Second Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbb{N}$.

Odd Even Mathematical Induction

Let $a_1 = 2$, $a_2 = 2$ $a_{n+2} = a_n + 1$

Prove that $a_n = \frac{1}{2}(n+1) + \frac{1}{4}\left[1 + (-1)^n\right]$.

Solution

Let P(n) be the proposition : $a_n = \frac{1}{2}(n+1) + \frac{1}{4}\left[1+(-1)^n\right]$.

For P(1), $a_1 = 1 = \frac{1}{2}(1+1) + \frac{1}{4}\left[1+(-1)^1\right]$

For P(2), $a_2 = 1 = \frac{1}{2}(2+1) + \frac{1}{4}\left[1+(-1)^2\right]$ $\therefore$ P(1) $\wedge$ P(2) is true.

Assume P(k) is true for some $k \in \mathbb{N}$. i.e. $a_k = \frac{1}{2}(k+1) + \frac{1}{4}\left[1+(-1)^k\right]$ …. (*)

For P(k + 2),

$a_{k+2} = a_k + 1 = \frac{1}{2}(k+1) + \frac{1}{4}\left[1+(-1)^k\right] + 1$, by (*)

$= \frac{1}{2}\left[(k+1)+1\right] + \frac{1}{4}\left[1+(-1)^{k+1}\right]$

$\therefore$ P(k + 2) is true.

$\therefore$ By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbb{N}$.

Backward Mathematical Induction

Let f(x) be a convex function defined on [a, b], i.e. $f(x_1) + f(x_2) \le 2f\left(\frac{x_1 + x_2}{2}\right)$ for all $x_1, x_2 \in [a, b]$.

For each positive integer n, consider the statement:

I(n) : If $x_i \in [a, b]$, i = 1, 2, …, n, then $f(x_1) + \ldots + f(x_n) \le nf\left(\frac{x_1 + \ldots + x_n}{n}\right)$.

(a) Prove by induction that $I(2^k)$ is true for every positive integer k.

(b) Prove that if I(n) $(n \ge 2)$ is true, then I(n-1) is true.

(c) Prove that I(n) is true for every positive integer n.

Solution

(a) I(n) : If $x_i \in [a, b]$, i = 1, 2, …, n, then $f(x_1) + \ldots + f(x_n) \le nf\left(\frac{x_1 + \ldots + x_n}{n}\right)$

For $I(2^1)$, since it is given that $f(x_1) + f(x_2) \le 2f\left(\frac{x_1 + x_2}{2}\right)$. $\therefore$ $I(2^1)$ is true.

Assume $I(2^k)$ is true. i.e. $f(x_1) + \ldots + f(x_{2^k}) \le 2^k f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right)$ ….(1)

For $I(2^{k+1})$,

$f(x_1) + \ldots + f(x_{2^k}) + f(x_{2^k+1}) + \ldots + f(x_{2^{k+1}})$

$\le 2^k f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + 2^k f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right) = 2^k\left[f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right)\right]$, by (1)

$= 2^k\left[f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right)\right] \le 2^k 2\left[f\left(\frac{1}{2}\left(\frac{x_1 + \ldots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right)\right)\right]$, by I(2)

$= 2^{k+1}\left[f\left(\frac{x_1 + \ldots + x_{2^k} + x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^{k+1}}\right)\right]$ $\therefore$ $I(2^{k+1})$ is true

(b) Assume I(n) is true $(n \ge 2)$,

i.e. $f(x_1)+\ldots+f(x_n) \le nf\left(\frac{x_1+\ldots+x_n}{n}\right) = nf\left(\frac{n-1}{n}\left(\frac{x_1+\ldots+x_{n-1}}{n-1}+\frac{x_n}{n-1}\right)\right)$

Put $x_n = \frac{x_1+\ldots+x_{n-1}}{n-1}$, then

$$f(x_1)+\ldots+f(x_{n-1})+f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right) \le nf\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right)$$

$f(x_1)+\ldots+f(x_{n-1}) \le (n-1)f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right)$ $\therefore$ I(n – 1) is also true.

(c) $\forall n \in \mathbb{N}$, $\exists$ ($k \in \mathbb{N}$ and $r \in \mathbb{N}$) such that $n = 2^k - r$.

<u>Spiral Mathematical Induction</u>

Given a sequence $\{a_n\}$ satisfying $a_{2m-1} = 3m(m-1)+1$ and $a_{2m} = 3m^2$, where $m \in \mathbb{N}$.

Let $S_n = \sum_{i=1}^{n} a_i$, prove that $\begin{cases} S_{2m-1} = \frac{1}{2}m(4m^2-3m+1) & \ldots.(1) \\ S_{2m} = \frac{1}{2}m(4m^2+3m+1) & \ldots.(2) \end{cases}$

Solution

Let P(m) be the proposition : $S_{2m-1} = \frac{1}{2}m(4m^2-3m+1)$

Q(m) be the proposition : $S_{2m} = \frac{1}{2}m(4m^2+3m+1)$

For P(1), $S_1 = a_1 = 1$ $\therefore$ (1) is true for $m = 1$.

Assume P(k) is true for some $k \in \mathbb{N}$., i.e. $S_{2k-1} = \frac{1}{2}k(4k^2-3k+1)$ …. (*)

(a) For Q(k), $S_{2k} = S_{2k-1} + a_{2k} = \frac{1}{2}k(4k^2-3k+1)+3k^2 = \frac{1}{2}k(4k^2+3k+1)$. $\therefore$ Q(k) is true.

(b) For P(k + 1),

$$S_{2k+1} = S_{2k} + a_{2k+1} = \frac{1}{2}k(4k^2+3k+1)+[3(k+1)k+1]$$

$$= \frac{1}{2}[(4k^3+12k^2+12k+4)-(3k^2+6k+3)+(k+1)]$$

$$= \frac{1}{2}[4(k+1)^3-3(k+1)^2+(k+1)] = \frac{1}{2}(k+1)[4(k+1)^2-3(k+1)+1]$$. $\therefore$ P(k + 1) is true.

Since (1) P(1) is true.

(2) P(k) is true $\Rightarrow$ Q(k) is true $\Rightarrow$ P(k + 1) is true

$\therefore$ By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbb{N}$.

Since (1) P(1) is true. $\Rightarrow$ Q(1) is true

(2) Q(k) is true $\Rightarrow$ P(k + 1) is true $\Rightarrow$ Q(k + 1) is true

$\therefore$ By the Principle of Mathematical Induction, Q(n) is true $\forall n \in \mathbb{N}$.

<u>Mathematical Induction with parameter</u>

Let $\quad f(a, 1) = \begin{cases} 1 & , \text{when } a = 1 \\ 0 & , \text{when } a > 1,\ a \in \mathbf{N} \end{cases}$.

and $\quad f(a, n+1) = \begin{cases} f(a,n) + 1 & , \text{when } a = 1 \\ f(a,n) + f(a-1,n) & , \text{when } a > 1,\ a \in \mathbf{N} \end{cases}$.

Prove that $\quad f(a,n) = \dfrac{n(n-1)\ldots(n-a+1)}{a!}$

Solution

Let P(n) be the proposition : $f(a,n) = \dfrac{n(n-1)\ldots(n-a+1)}{a!}$ …. (1)

(1) For P(1), there are two cases:

When a = 1, L.H.S. = f(1, 1) = 1. R.H.S. = $\dfrac{1}{1!} = 1$

When a > 1, L.H.S. = f(a, 1) = 0. R.H.S. = $\dfrac{1(n-1)\ldots(1-a+1)}{a!} = 0$. $\therefore$ P(1) is true.

(2) Assume P(k) is true for some $k \in \mathbb{N}$., i.e. $f(a,k) = \dfrac{k(k-1)\ldots(k-a+1)}{a!}$ …. (2)

For P(k + 1), there are also two cases:

When a = 1, L.H.S. = f(a, k + 1) = f(a, k) + 1 = $\dfrac{k}{1!} + 1 = k + 1 = \dfrac{k+1}{1!}$ = R.H.S.

When a > 1, L.H.S. = f(a,k) +f(a – 1, k)

$$= \frac{k(k-1)\ldots(k-a+1)}{a!} + \frac{k(k-1)\ldots(k-a+2)}{(a-1)!}$$, by (2), f(a,k) and f(a – 1, k) hold .

$$= \frac{k(k-1)\ldots(k-a+2)}{a!}\left[(k-a+1)+a\right]$$

$$= \frac{(k+1)k(k-1)\ldots(k-a+2)}{a!} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbb{N}$.

<u>Comment</u> If the proposition with natural number n contains a parameter a , then we need to apply mathematical induction for all values of a .

Double Mathematical Induction

Prove that the number of non-negative integral solution sets of the equation

$x_1 + x_2 + \dots + x_m = n \quad , \quad m , n \in \mathbb{N}$.

is $\quad f(m, n) = \dfrac{(n+m-1)!}{n!(m-1)!} \quad \dots. \quad (1)$

Solution

Let P(m, n) be the given proposition.

(a) For P(1, n), The only non-negative integral solution set of the equation $x_1 = n$ is only itself .

In (1) , $f(1, n) = \dfrac{(n+1-1)!}{n!(1-1)!} = 1$.

$\therefore$ P(1, n) is true .

For P(m, 1), The non-negative integral solution sets of the equation

$x_1 + x_2 + \dots + x_m = 1$

are (1, 0, 0 …, 0) , (0, 1, 0, …) , …, (0, 0, 0, …, 1) .

There are m sets of solution altogether.

In (1), $f(m, 1) = \dfrac{(1+m-1)!}{1!(m-1)!} = m$.

$\therefore$ P(m, 1) is true .

(b) Assume P(m, n+1) and P(m + 1, n) are true for some $m , n \in \mathbb{N}$ . . i.e

the number of non-negative integral solution sets of the equations :

$x_1 + x_2 + \dots + x_m = n + 1 \quad \dots. \quad (2)$

$x_1 + x_2 + \dots + x_m + x_{m+1} = n \quad \dots. \quad (3)$

are $f(m, n+1) = \dfrac{(n+m)!}{(n+1)!(m-1)!}$ and $f(m+1, n) = \dfrac{(n+m)!}{n!m!}$ respectively .

For P(m+1, n+1), The non-negative integral solution sets of the equation :

$x_1 + x_2 + \dots + x_m + x_{m+1} = n + 1 \quad \dots. \quad (4)$

may be divided into two parts : $x_{m+1} = 0$ or $x_{m+1} > 0$.

(i) For $x_{m+1} = 0$, equation (4) becomes equation (2), and the number of non-negative integral solution sets is $f(m, n+1) = \dfrac{(n+m)!}{(n+1)!(m-1)!}$.

(ii) For $x_{m+1} > 0$, replace x_{m+1} by $x_{m+1} + 1$ and equation (4) becomes:

$x_1 + x_2 + \dots + x_m + x_{m+1} = n$, and the number of non-negative integral solution sets is $f(m+1, n) = \dfrac{(n+m)!}{n!m!}$.

$\therefore$ The total number of non-negative integral solution sets is

$$\frac{(n+m)!}{(n+1)!(m-1)!} + \frac{(n+m)!}{n!m!} = \frac{(n+m)!}{(n+1)!m!}[(n+1)+m] = \frac{[(n+1)+(m+1)-1]}{(n+1)![(m+1)-1]!} .$$

$\therefore$ P(m+1, n+1) is also true .

$\therefore$ By the Principle of Mathematical Induction, P(m, n) is true $\forall\ m, n \in \mathbb{N}$.