1.10 Β. ΠΡΟΣΘΕΣΗ – ΑΦΑΙΡΕΣΗ ΡΗΤΩΝ ΠΑΡΑΣΤΑΣΕΩΝ

Ασκήσεις σχ. βιβλίου σελίδων 80-81

Ερωτήσεις κατανόησης

1 .

Να χαρακτηρίσετε τις παρακάτω ισότητες με (Σ) αν είναι σωστές και με (Λ) αν είναι λανθασμένες.

α) $\frac{x}{x+1} + \frac{1}{x+1} = 1$ (Σ)

β) $\frac{1}{x} + \frac{1}{y} = \frac{2}{x+y}$ (Λ)

γ) $\frac{\alpha+4}{\alpha} - \frac{4}{\alpha} = 1$ (Σ)

δ) $\frac{\alpha+\beta}{\alpha-\beta} + \frac{\alpha+\beta}{\beta-\alpha} = 0$ (Σ)

ε) $1 + \frac{x}{\omega} = \frac{1+x}{\omega}$ (Λ)

στ) $\frac{\alpha}{x} - \frac{\alpha+2}{x} = \frac{2}{x}$ (Λ)

Προτεινόμενη λύση

α) $\frac{x}{x+1} + \frac{1}{x+1} = \frac{x+1}{x+1} = 1$ άρα (Σ)

β) Προφανώς (Λ)

γ) $\frac{\alpha+4}{\alpha} - \frac{4}{\alpha} = \frac{\alpha+4-4}{\alpha} = \frac{\alpha}{\alpha} = 1$ άρα (Σ)

δ) $\frac{\alpha+\beta}{\alpha-\beta} + \frac{\alpha+\beta}{\beta-\alpha} = \frac{\alpha+\beta}{\alpha-\beta} - \frac{\alpha+\beta}{\alpha-\beta} = 0$ άρα (Σ)

ε) Προφανώς (Λ)

στ) $\frac{\alpha}{x} - \frac{\alpha+2}{x} = \frac{\alpha-\alpha-2}{x} = \frac{-2}{x}$ άρα (Λ)

2.

Ένας μαθητής έγραψε τις παρακάτω ισότητες και ο καθηγητής του είπε ότι σε κάποιο σημείο έκανε ένα λάθος. Μπορείτε να εντοπίσετε το λάθος αυτό ;

α) $\frac{\alpha}{\alpha-\beta} + \frac{\beta}{\beta-\alpha} = \frac{\alpha}{\alpha-\beta} - \frac{\beta}{\alpha-\beta} = \frac{\alpha-\beta}{\alpha-\beta} = 1$

β) $\frac{3x+2}{x+1} - \frac{2x-1}{x+1} = \frac{3x+2-2x-1}{x+1} = \frac{x+1}{x+1} = 1$

Προτεινόμενη λύση

Που έχει γίνει το λάθος φαίνεται παρακάτω

$$\frac{3x+2}{x+1} - \frac{2x-1}{x+1} = \frac{3x+2-(2x-1)}{x+1} = \frac{3x+2-2x+1}{x+1} = \frac{x+3}{x+1}$$

3.

Να συμπληρώσετε τις ισότητες

α) $\frac{x}{x+6} - \frac{x}{x+6} = 0$ **β)** $\frac{x}{x+6} + \frac{6}{x+6} = 1$ **γ)** $\frac{x}{x+1} + \frac{x}{x+1} = \frac{2x}{x+1}$

δ) $\frac{6}{x+2} - \frac{5}{x+2} = \frac{1}{x+2}$ **ε)** $\frac{2x-1}{x} + \frac{1}{x} = 2$ **στ)** $\frac{3x+8}{x} - \frac{8}{x} = 3$

Ασκήσεις

1.

Να υπολογίσετε τις παραστάσεις

α) $\frac{1}{x} + \frac{1}{y}$ **β)** $\frac{3}{x+1} - \frac{2}{x}$ **γ)** $\frac{1}{y^2} - \frac{1}{y}$ **δ)** $\frac{1}{\omega^2} - \frac{2}{\omega^2+1}$

Προτεινόμενη λύση

α)

$$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{y+x}{xy}$$

β)

$$\frac{3}{x+1} - \frac{2}{x} = \frac{3x}{x(x+1)} - \frac{2(x+1)}{x(x+1)} = \frac{3x-2(x+1)}{x(x+1)} = \frac{3x-2x-2}{x(x+1)} = \frac{x-2}{x(x+1)}$$

γ)

$$\frac{1}{y^2} - \frac{1}{y} = \frac{1}{y^2} - \frac{y}{y^2} = \frac{1-y}{y^2}$$

δ)

$$\frac{1}{\omega^2} - \frac{2}{\omega^2+1} = \frac{\omega^2+1}{\omega^2(\omega^2+1)} - \frac{2\omega^2}{\omega^2(\omega^2+1)} = \frac{\omega^2+1-2\omega^2}{\omega^2(\omega^2+1)} = \frac{1-\omega^2}{\omega^2(\omega^2+1)}$$

2.

Να υπολογίσετε τις παραστάσεις

α) $\dfrac{2x}{2x-6} - \dfrac{3}{x-3}$ **β)** $\dfrac{y-6}{y^2+2y} - \dfrac{4}{y+2}$ **γ)** $\dfrac{3\omega+6}{\omega^2-4} - \dfrac{4}{2\omega-4}$

δ) $\dfrac{1}{2x+12} + \dfrac{x}{36-x^2}$ **ε)** $\dfrac{9x}{x^2-x\omega} + \dfrac{3\omega}{\omega^2-x\omega}$ **στ)** $\dfrac{\alpha+7}{\alpha^2+4\alpha+3} - \dfrac{3}{\alpha+1}$

Προτεινόμενη λύση

α)

$$\frac{2x}{2x-6} - \frac{3}{x-3} = \frac{2x}{2(x-3)} - \frac{3}{x-3} = \frac{x}{x-3} - \frac{3}{x-3} = \frac{x-3}{x-3} = 1$$

β)

$$\frac{y-6}{y^2+2y} - \frac{4}{y+2} = \frac{y-6}{y(y+2)} - \frac{4}{y+2} =$$

$$= \frac{y-6}{y(y+2)} - \frac{4y}{y(y+2)} = \frac{y-6-4y}{y(y+2)} =$$

$$= \frac{-3y-6}{y(y+2)} = \frac{-3(y+2)}{y(y+2)} = -\frac{3}{y}$$

γ)

$$\frac{3\omega+6}{\omega^2-4} - \frac{4}{2\omega-4} = \frac{3(\omega+2)}{(\omega-2)(\omega+2)} - \frac{4}{2(\omega-2)} = \frac{3}{\omega-2} - \frac{2}{\omega-2} = \frac{1}{\omega-2}$$

δ)

$$\frac{1}{2x+12} + \frac{x}{36-x^2} = \frac{1}{2(x+6)} + \frac{x}{(6-x)(6+x)} =$$

$$= \frac{6-x}{2(x+6)(6-x)} + \frac{2x}{(6-x)(6+x)} =$$

$$= \frac{6-x+2x}{2(x+6)(6-x)} = \frac{6+x}{2(x+6)(6-x)} = \frac{1}{2(6-x)}$$

ε)

$$\frac{9x}{x^2-x\omega} + \frac{3\omega}{\omega^2-x\omega} = \frac{9x}{x(x-\omega)} + \frac{3\omega}{\omega(\omega-x)} =$$

$$= \frac{9x}{x(x-\omega)} - \frac{3\omega}{\omega(x-\omega)} =$$

$$= \frac{9x\omega}{x\omega(x-\omega)} - \frac{3\omega x}{x\omega(x-\omega)} =$$

$$= \frac{6x\omega}{x\omega(x-\omega)} = \frac{6}{x-\omega}$$

στ)

$$\frac{\alpha+7}{\alpha^2+4\alpha+3} - \frac{3}{\alpha+1} = \frac{\alpha+7}{(\alpha+1)(\alpha+3)} - \frac{3}{\alpha+1} =$$

$$= \frac{\alpha+7}{(\alpha+1)(\alpha+3)} - \frac{3(\alpha+3)}{(\alpha+1)(\alpha+3)} =$$

$$= \frac{\alpha+7-3(\alpha+3)}{(\alpha+1)(\alpha+3)} =$$

$$= \frac{\alpha+7-3\alpha-9}{(\alpha+1)(\alpha+3)} = \frac{-2\alpha-2}{(\alpha+1)(\alpha+3)} = \frac{-2(\alpha+1)}{(\alpha+1)(\alpha+3)} = -\frac{2}{\alpha+3}$$

3.

Να απλοποιήσετε τα κλάσματα

α) $\dfrac{x-\frac{1}{x}}{1+\frac{1}{x}}$ **β)** $\dfrac{y-2+\frac{1}{y}}{y-\frac{1}{y}}$ **γ)** $\dfrac{\omega+1+\frac{1}{\omega}}{1-\frac{1}{\omega^3}}$ **δ)** $\dfrac{\frac{1}{\alpha}-\frac{1}{\beta}}{\frac{\beta}{\alpha}-\frac{\alpha}{\beta}}$

Προτεινόμενη λύση

α)

$$\frac{x-\frac{1}{x}}{1+\frac{1}{x}} = \frac{\frac{x^2}{x}-\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}} = \frac{\frac{x^2-1}{x}}{\frac{x+1}{x}} = \frac{x(x^2-1)}{x(x+1)} = \frac{(x-1)(x+1)}{x+1} = x-1$$

β)

$$\frac{y-2+\frac{1}{y}}{y-\frac{1}{y}} = \frac{\frac{y^2}{y}-\frac{2y}{y}+\frac{1}{y}}{\frac{y^2}{y}-\frac{1}{y}} = \frac{\frac{y^2-2y+1}{y}}{\frac{y^2-1}{y}} = \frac{y(y^2-2y+1)}{y(y^2-1)}$$

$$= \frac{(y-1)^2}{(y-1)(y+1)} = \frac{y-1}{y+1}$$

γ)

$$\frac{\omega+1+\frac{1}{\omega}}{1-\frac{1}{\omega^3}} = \frac{\frac{\omega^2}{\omega}+\frac{\omega}{\omega}+\frac{1}{\omega}}{\frac{\omega^3}{\omega^3}-\frac{1}{\omega^3}} = \frac{\frac{\omega^2+\omega+1}{\omega}}{\frac{\omega^3-1}{\omega^3}} = \frac{\omega^3(\omega^2+\omega+1)}{\omega(\omega^3-1)} =$$

$$= \frac{\omega^3(\omega^2+\omega+1)}{\omega(\omega-1)(\omega^2+\omega+1)} = \frac{\omega^2}{\omega-1}$$

δ)

$$\frac{\frac{1}{\alpha}-\frac{1}{\beta}}{\frac{\beta}{\alpha}-\frac{\alpha}{\beta}} = \frac{\frac{\beta}{\alpha\beta}-\frac{\alpha}{\alpha\beta}}{\frac{\beta^2}{\alpha\beta}-\frac{\alpha^2}{\alpha\beta}} = \frac{\frac{\beta-\alpha}{\alpha\beta}}{\frac{\beta^2-\alpha^2}{\alpha\beta}} = \frac{\alpha\beta(\beta-\alpha)}{\alpha\beta(\beta^2-\alpha^2)} = \frac{\beta-\alpha}{(\beta-\alpha)(\alpha+\beta)} = \frac{1}{\alpha+\beta}$$

4.

Να υπολογίσετε τις παραστάσεις

α) $\dfrac{x-2}{x} + \dfrac{4}{x-2} - \dfrac{8}{x^2-2x}$ **β)** $\dfrac{3}{x+2y} - \dfrac{2}{x-2y} + \dfrac{2x+16y}{x^2-4y^2}$

γ) $\dfrac{y^2-6}{y^2-5y+6} - \dfrac{2}{y-2} + \dfrac{3}{y-3}$ **δ)** $\dfrac{x^2}{x-y} + \dfrac{y^2}{x+y} - \dfrac{2xy^2}{x^2-y^2}$

Προτεινόμενη λύση

α)

$$\frac{x-2}{x} + \frac{4}{x-2} - \frac{8}{x^2-2x} = \frac{x-2}{x} + \frac{4}{x-2} - \frac{8}{x(x-2)} =$$

$$= \frac{(x-2)^2}{x(x-2)} + \frac{4x}{x(x-2)} - \frac{8}{x(x-2)} =$$

$$= \frac{(x-2)^2+4x-8}{x(x-2)} =$$

$$= \frac{x^2-4x+4+4x-8}{x(x-2)} =$$

$$= \frac{x^2-4}{x(x-2)} = \frac{(x-2)(x+2)}{x(x-2)} = \frac{x+2}{x} \quad \text{με } x \neq 0 \text{ και } x \neq 2$$

β)

$$\frac{3}{x+2y} - \frac{2}{x-2y} + \frac{2x+16y}{x^2-4y^2} = \frac{3}{x+2y} - \frac{2}{x-2y} + \frac{2x+16y}{(x-2y)(x+2y)} =$$

$$= \frac{3(x-2y)}{(x-2y)(x+2y)} - \frac{2(x+2y)}{(x-2y)(x+2y)} + \frac{2x+16y}{(x-2y)(x+2y)} =$$

$$= \frac{3(x-2y)-2(x+2y)+2x+16y}{(x-2y)(x+2y)} =$$

$$= \frac{3x-6y-2x-4y+2x+16y}{(x-2y)(x+2y)} =$$

$$= \frac{3x+6y}{(x-2y)(x+2y)} =$$

$$= \frac{3(x+2y)}{(x-2y)(x+2y)} = \frac{3}{x-2y} \quad \text{με } x \neq 2y \text{ και } x \neq -2y$$

γ)

$$\frac{y^2-6}{y^2-5y+6} - \frac{2}{y-2} + \frac{3}{y-3} = \frac{y^2-6}{(y-2)(y-3)} - \frac{2}{y-2} + \frac{3}{y-3} =$$

$$= \frac{y^2-6}{(y-2)(y-3)} - \frac{2(y-3)}{(y-2)(y-3)} + \frac{3(y-2)}{(y-2)(y-3)} =$$

$$= \frac{y^2-6-2(y-3)+3(y-2)}{(y-2)(y-3)}$$

$$= \frac{y^2-6-2y+6+3y-6}{(y-2)(y-3)} = \frac{y^2+y-6}{(y-2)(y-3)}$$

$$= \frac{(y+3)(y-2)}{(y-2)(y-3)} = \frac{y+3}{y-3} \quad \text{με} \quad y \neq 2 \quad \text{και} \quad y \neq 3$$

δ)

$$\frac{x^2}{x-y} + \frac{y^2}{x+y} - \frac{2xy^2}{x^2-y^2} = \frac{x^2}{x-y} + \frac{y^2}{x+y} - \frac{2xy^2}{(x+y)(x-y)} =$$

$$= \frac{x^2(x+y)}{(x+y)(x-y)} + \frac{y^2(x-y)}{(x+y)(x-y)} - \frac{2xy^2}{(x+y)(x-y)} =$$

$$= \frac{x^2(x+y)+y^2(x-y)-2xy^2}{(x+y)(x-y)} =$$

$$= \frac{x^3+x^2y+y^2x-y^3-2xy^2}{(x+y)(x-y)} =$$

$$= \frac{x^3+x^2y-y^2x-y^3}{(x+y)(x-y)} =$$

$$= \frac{x^2(x+y)-y^2(x+y)}{(x+y)(x-y)} =$$

$$= \frac{(x+y)(x^2-y^2)}{(x+y)(x-y)}$$

$$= \frac{(x+y)(x-y)(x+y)}{(x+y)(x-y)} = x+y \quad \text{με } x \neq y \text{ και } x \neq -y$$

5.

Να υπολογίσετε τις παραστάσεις

α) $\left(\dfrac{x+3}{2x+1}-\dfrac{x}{2x-1}\right)\left(1+\dfrac{1}{4x-3}\right)$ **β)** $\left[\dfrac{x+3}{x^2-1}+\dfrac{x-3}{(x-1)^2}\right]:\dfrac{x^2-3}{(x-1)^2}$

γ) $\left(1-\dfrac{2\alpha\beta}{\alpha^2+\beta^2}\right)\left(\dfrac{\alpha}{\beta}+\dfrac{\alpha+\beta}{\alpha-\beta}\right)$ **δ)** $\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}-1\right):\left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right)$

Προτεινόμενη λύση

α)

$$\left(\frac{x+3}{2x+1}-\frac{x}{2x-1}\right)\left(1+\frac{1}{4x-3}\right)=$$

$$=\left(\frac{(x+3)(2x-1)}{(2x+1)(2x-1)}-\frac{x(2x+1)}{(2x+1)(2x-1)}\right)\left(\frac{4x-3}{4x-3}+\frac{1}{4x-3}\right)=$$

$$=\frac{(x+3)(2x-1)-x(2x+1)}{(2x+1)(2x-1)}\cdot\frac{4x-3+1}{4x-3}=$$

$$=\frac{2x^2-x+6x-3-2x^2-x}{(2x+1)(2x-1)}\cdot\frac{4x-2}{4x-3}=$$

$$=\frac{4x-3}{(2x+1)(2x-1)}\cdot\frac{2(2x-1)}{4x-3}=\frac{2}{2x+1}\quad \text{με } x\neq\frac{1}{2}\ \text{ και } x\neq-\frac{1}{2}\ \text{ και } x\neq\frac{3}{4}$$

β)

$$\left[\frac{x+3}{x^2-1}+\frac{x-3}{(x-1)^2}\right]:\frac{x^2-3}{(x-1)^2}=\left[\frac{x+3}{(x+1)(x-1)}+\frac{x-3}{(x-1)^2}\right]\cdot\frac{(x-1)^2}{x^2-3}=$$

$$=\left[\frac{(x+3)(x-1)}{(x+1)(x-1)^2}+\frac{(x-3)(x+1)}{(x+1)(x-1)^2}\right]\cdot\frac{(x-1)^2}{x^2-3}=$$

$$=\left[\frac{(x+3)(x-1)+(x-3)(x+1)}{(x+1)(x-1)^2}\right]\cdot\frac{(x-1)^2}{x^2-3}=$$

$$=\frac{x^2-x+3x-3+x^2+x-3x-3}{(x+1)(x-1)^2}\cdot\frac{(x-1)^2}{x^2-3}=$$

$$=\frac{2x^2-6}{(x+1)(x-1)^2}\cdot\frac{(x-1)^2}{x^2-3}=$$

$$=\frac{2(x^2-3)(x-1)^2}{(x+1)(x-1)^2(x^2-3)}=$$

$$=\frac{2}{x+1}\quad \text{με } x\neq 1 \text{ και } x\neq -1 \text{ και } x\neq\sqrt{3}\ \text{ και } x\neq-\sqrt{3}$$

γ)

$$\left(1-\frac{2\alpha\beta}{\alpha^2+\beta^2}\right)\left(\frac{\alpha}{\beta}+\frac{\alpha+\beta}{\alpha-\beta}\right)=\frac{\alpha^2+\beta^2-2\alpha\beta}{\alpha^2+\beta^2}\left[\frac{\alpha(\alpha-\beta)}{\beta(\alpha-\beta)}+\frac{\beta(\alpha+\beta)}{\beta(\alpha-\beta)}\right]=$$

$$=\frac{\alpha^2+\beta^2-2\alpha\beta}{\alpha^2+\beta^2}\cdot\frac{\alpha^2-\alpha\beta+\alpha\beta+\beta^2}{\beta(\alpha-\beta)}=$$

$$= \frac{(\alpha-\beta)^2(\alpha^2+\beta^2)}{(\alpha^2+\beta^2)\beta(\alpha-\beta)} = \frac{\alpha-\beta}{\beta} \quad \text{με } \beta \neq 0 \text{ και } \alpha \neq \beta$$

δ)

$$\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 1\right) : \left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) = \frac{\alpha^2+\beta^2-\alpha\beta}{\alpha\beta} : \frac{\alpha^3+\beta^3}{\alpha\beta} =$$

$$= \frac{\alpha^2+\beta^2-\alpha\beta}{\alpha\beta} \cdot \frac{\alpha\beta}{\alpha^3+\beta^3} =$$

$$= \frac{(\alpha^2+\beta^2-\alpha\beta)\alpha\beta}{\alpha\beta(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)} =$$

$$= \frac{1}{\alpha+\beta} \quad \text{με } \alpha \neq 0 \text{ και } \beta \neq 0 \text{ και } \alpha \neq -\beta$$

6.

α) Να αποδείξετε ότι $\frac{x^3-y^3}{x-y} + xy = (x+y)^2$

β) Να υπολογίσετε την παράσταση $\frac{56^3-44^3}{12} + 56 \cdot 44$

Προτεινόμενη λύση

α)

$$\frac{x^3-y^3}{x-y} + xy = \frac{(x-y)(x^2+xy+y^2)}{x-y} + xy = x^2 + xy + y^2 + xy =$$

$$= x^2 + 2xy + y^2 =$$

$$= (x+y)^2$$

β)

Η ζητούμενη παράσταση προκύπτει από το πρώτο μέλος του (α) για $x = 56$ και $y = 44$.

Επομένως $\frac{56^3-44^3}{12} + 56 \cdot 44 = (56+44)^2 = 100^2 = 10000$

7.

α) Αν $A = \dfrac{2x}{x^2+1}$ και $B = \dfrac{x^2-1}{x^2+1}$, να αποδείξετε ότι $A^2 + B^2 = 1$

β) Να αποδείξετε ότι οι αριθμοί 1 , $\dfrac{200}{10001}$, $\dfrac{9999}{10001}$ αποτελούν μήκη πλευρών ορθογωνίου τριγώνου.

Προτεινόμενη λύση

α)

$$A^2 + B^2 = \left(\frac{2x}{x^2+1}\right)^2 + \left(\frac{x^2-1}{x^2+1}\right)^2 = \frac{4x^2}{(x^2+1)^2} + \frac{x^4-2x^2+1}{(x^2+1)^2} =$$

$$= \frac{4x^2+x^4-2x^2+1}{(x^2+1)^2} =$$

$$= \frac{x^4+2x^2+1}{(x^2+1)^2} = \frac{(x^2+1)^2}{(x^2+1)^2} = 1$$

β)

Αν $x = 100$ τότε $\dfrac{200}{10001} = \dfrac{2x}{x^2+1}$ και $\dfrac{9999}{10001} = \dfrac{x^2-1}{x^2+1}$

Επομένως , σύμφωνα με το (α), έχουμε $\left(\dfrac{200}{10001}\right)^2 + \left(\dfrac{9999}{10001}\right)^2 = 1^2$

Άρα η τριάδα 1 , $\dfrac{200}{10001}$, $\dfrac{9999}{10001}$ αποτελεί τριάδα πλευρών ορθογωνίου τριγώνου με υποτείνουσα το 1.